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Dynamic systems with high damping rubber: Nonlinear behaviour and linear approximation

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SUMMARY

High damping rubber (HDR) shows a quite complex constitutive behaviour, which is nonlinear with respect to strain and is dependent on the strain rate. In addition, it exhibits a transient response during which the material properties change (scragging or more generally the Mullins effect). A number of recent works were dedicated to analysing and modelling material behaviour. This paper studies the nonlinear dynamics of systems with restoring force produced by HDR-based devices in order to propose a procedure to define equivalent linear models considering both transient and stationary behaviours. The reliability of these linear models is tested by evaluating the upper and lower bounds of the seismic response of a structural system equipped with HDR-based devices (structural system with dissipative bracings and isolated systems). Copyright © 2008 John Wiley & Sons, Ltd.

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KEY WORDS: high damping rubber; dissipating devices; seismic isolation; nonlinear behaviour; equivalent linear models

1. INTRODUCTION

In recent years, high damping rubbers (HDRs) have been extensively employed in seismic bearings and dissipation devices introduced in structural systems in order to reduce seismic effects. In the first case they allow the supports to deform with a large dissipation and re-centering capacity [1]. In the second case they improve the stiffness and dissipation capacity of frame structures (new or existing) when utilized in dissipating braces [2–5].

HDR is obtained by adding a filler to the natural rubber in order to improve some properties such as the strength, the stiffness of the rubber as well as its dissipation capacity. On the other hand the filler introduces an additional source of nonlinearity in the rubber behaviour. In particular, the transient behaviour tends to rise up and there is a progressive loss of rigidity due to the damage of

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the internal microstructure. In the case of a cyclical test with constant amplitude this phenomenon involves the first load cycles and progresses through to stabilization of the damage (the effect is known as 'scragging'). In general, the damage involves a strong dependence of the material response on the load history and, particularly, on the maximum deformation experienced along the path. This phenomenon is known as the Mullins effect [6–8]. It should also be noted that the initial properties of the material are recovered in a sufficiently short space of time [1, 9] so that variations of the stiffness and dissipation capacity due to the Mullins effect always influence the seismic response.

In recent years, the use of HDR in the production of dissipation and isolation devices oriented research works towards the definition of appropriate constitutive laws to describe the response in the case of cyclical shear loads with or without compression. Papers that aim at describing the seismic behaviour of HDR devices usually propose constitutive laws based on simple models modified to take into account the strongly nonlinear strain dependence and the strain rate dependence [10–12]. Few works have attempted to model even the process-dependent behaviour [13]. A test program to fully investigate the pure shear behaviour in a range of strain and strain rates in order to mitigate seismic effects is described in [9], which also proposes an analytical model based on a rheological thermodynamically compatible approach, in which internal variables were introduced to describe the inelastic phenomena and the damaging effects.

Previous models may be used to perform nonlinear computations, even if they are usually quite complex, require the calibration of many parameters and are not always available in commercial computational codes. On the other hand the description of the material behaviour by linear models [14, 15] is a simplified approach usually adopted in practical applications, which is accepted by more recent guidelines [16–18] for both a response spectrum analysis or a linear time history analysis. This approach may be convenient because very few parameters need to be identified from the experimental data and structural analysis may be performed by means of very common commercial codes.

In the linear approach the transient behaviour of the rubber is generally neglected by practical codes [16, 17], or imprecise information regarding the procedure to follow in order to take it into account is furnished [18]. The scope of this paper is to study the dynamic behaviour of nonlinear systems with a restoring force provided by HDR devices and, on the bases of this, to define linear equivalence criteria (consequently linear model parameters), able to furnish the most similar linear dynamic behaviour. In particular, two reference situations (motions) are considered. The first refers to the stationary behaviour under sinusoidal excitations, with the aim of providing a description of the stationary behaviour of the system and verifying the procedures suggested by the codes. The second concerns the transient response of the system subjected to an impulsive excitation, with the aim of furnishing a description of the initial part of the response, which strongly influences the seismic response. Four systems with different dynamic properties, showing a maximum response in the range period of 0.5–3.0 s, and three different shear strain values up to 200% are considered. Results may be of interest for purposes of structural systems with different stiffness (from frames with dissipating braces to isolated structures), while the strain range considered covers the maximum values usually adopted in the design. Numerical results are obtained by adopting the rheological thermodynamically compatible model proposed by the authors [9], which refers to rubber with an intermediate stiffness value.

Finally, the two-step linear model proposed is used to perform linear time history analyses under selected ground motions in order to check the reliability of the proposed linear models in evaluating the upper and lower bounds of the seismic response (maximum displacements and forces) of the

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single degree-of-freedom (S-DoF) systems studied. The results obtained are satisfactory and suggest that the linear models may also be useful for the analysis of multiple degree-of-freedom systems, based on linear time histories or simplified procedures, such as the response spectrum method.

2. NONLINEAR DYNAMICAL SYSTEM

The S-DoF dynamic system considered consists of a mass m and a dissipating device, based on HDR subjected to shear strain, which provides the restoring force. The constitutive behaviour of rubber is described by means of the model proposed in [9]. This is a thermodynamically consistent rheological model in which the free energy per unit volume $\varphi_{\mathrm{d}}(\gamma;\alpha_i)$ is expressed by means of the *material state* (γ,α_i) consisting of the shear strain γ and a set of internal variables α_i describing the inelastic phenomena related to Mullins effect and viscosity. The shear stress may be deduced by the derivative of the free energy:

$$\tau_{\mathrm{d}}=\frac{\partial\varphi_{\mathrm{d}}}{\partial\gamma} \tag{1}$$

and the evolution of the internal variables is furnished by

$$\dot{\alpha}_i=g(\gamma,\alpha_i,\eta) \tag{2}$$

once the state and the *process* $\eta=\dot{\gamma}$ are known (the superposed dot denotes the time derivative). The dissipated power per unit volume w_{d} may be obtained from the derivative with respect to the internal variables (repeated indexes denote summation)

$$w_{\mathrm{d}}=\frac{\partial\varphi_{\mathrm{d}}}{\partial\alpha_i}\dot{\alpha}_i \tag{3}$$

The *state* of the dynamical system is consequently described by the vector $\mathbf{x}=[u,v;\alpha_i]$, where u and v are the displacement and the velocity of the mass. The rubber strain $\gamma=\beta u/h$ and the strain rate $\eta=\beta v/h$ are proportional to displacement and velocity and may be related by introducing the geometric parameter β depending on the type of connection between the mass and the device. The parameter h represents the thickness of the rubber layer.

The restoring force per unit mass f_{d} may be expressed in the form

$$f_{\mathrm{d}}=\frac{\beta A}{m}\tau_{\mathrm{d}}\left(\frac{\beta u}{h},\frac{\beta v}{h},\alpha_i\right) \tag{4}$$

where A is the area of the HDR layer in the device. The evolution law of the dynamic system may be expressed as follows:

$$\dot{\mathbf{x}}=\begin{bmatrix}\dot{u}\\ \dot{v}\\ \dot{\alpha}_i\end{bmatrix}=\begin{bmatrix} v\\ -f_{\mathrm{d}}\left(\frac{\beta u}{h},\alpha_i;\frac{\beta v}{h}\right)+f_{\mathrm{e}}\\ g_i\left(\frac{\beta u}{h},\alpha_i;\frac{\beta v}{h}\right)\end{bmatrix}=\mathbf{A}(\mathbf{x}) \tag{5}$$

where f_{e} is the external force per unit mass.

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3. LINEAR EQUIVALENT SYSTEM

3.1. Equivalence conditions

In the system considered, only the relationship between the restoring force and the mass displacement is nonlinear so that the search for a simplified model may be reduced to the device only.

The equivalent linear system considered consists of the same mass m of the nonlinear system and a restoring force produced by an elastic spring and a dashpot placed in parallel. Based on the above, the system is completely defined once the two parameters k and c, describing, respectively, the spring stiffness for unity of mass and the viscous constant of the dashpot, again, for unity of mass, are assigned. The dissipation capacity of the linear system may also be expressed through the viscous damping coefficient ξ, defined as

$$\xi = \frac{c}{2\omega} \tag{6}$$

where $\omega = \sqrt{k}$ is the natural frequency of the linear nondamped system. Consequently, the restoring force of the linear system is

$$f_{\mathrm{d}}^{\mathrm{L}} = ku + 2\xi\omega v \tag{7}$$

The evolution law of the linear system state $\mathbf{x}^{\mathrm{L}} = [u, v]$ has the following form:

$$\dot{\mathbf{x}}^{\mathrm{L}} = \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -f_{\mathrm{d}}^{\mathrm{L}}(u, v) + f_{\mathrm{e}} \end{bmatrix} \tag{8}$$

A precise definition of the 'dynamic situation', where equivalence is required, is necessary to define a linear dynamic system that is 'equivalent' to a nonlinear one. For this reason, two motions, related to the linear and nonlinear systems, respectively, must be selected to apply the equivalent conditions. Once the corresponding motions are selected, two equivalence conditions must be applied in order to obtain the two parameters k and ξ, which characterize the linear system. It follows that different choices may be made in selecting the dynamic situations and equivalence conditions.

The nonlinear system considered in this work shows notable differences between the responses exhibited at the initial stage of motion, influenced by the Mullins effect, and the successive stage, when the internal damage reduces stiffness and the dissipative properties. Two different linear models are consequently determined in order to describe the two limit situations.

One model intends to describe the stationary behaviour of the material at the assigned strain and the strain rate, once Mullins effect is over. This considers the stationary response under sinusoidal external excitation. The maximum periodic motions observed in the linear and nonlinear systems were selected as corresponding dynamic situations.

The second model aims at describing the part of the motion strongly influenced by the Mullins effect and considers the response under impulsive excitation. The initial parts of the motions of linear and nonlinear systems were selected as corresponding dynamic situations.

In order to obtain coherent linear approximations, the equivalence conditions were chosen so that they could be applied to different motions. Among the possible choices, the equivalence conditions that would provide the most satisfactory results in the general description of the behaviour of the original dynamic system were preferred. The following sections illustrate the procedures followed in both cases and the results obtained.

DOI: 10.1002/eqe

3.2. Harmonic response

In this section, the stationary response of the nonlinear system subjected to sinusoidal external forces is adopted to define equivalent linear models at different levels of the strain.

In the tested range of external actions the system shows a transient response and attains a stationary behaviour after a certain number of cycles. The response related to an external action with period T is considered to be stationary at the instant t when the difference between the state history observed in the current period and the state history observed in the previous period is sufficiently small. It should also be observed that the dynamical system analysed exhibits a process-dependent response. In this case, contrary to what happens in the case of elastic systems, the stationary response depends on the initial conditions. In this section the aim is to characterize the response under seismic events acting on the system where the state variables are initially zero. The external force may be expressed as

$$f_{\mathrm{e}}(t) = f_0 \sin(2\pi t/T) \tag{9}$$

where f_0 is the amplitude of the force per unit mass and T is its period. The equivalent linear system is obtained by adopting the following rules:

(i) The periodic motions of the two systems at the frequencies at which the maximum displacements are attained are selected as similar dynamic situations to apply the equivalence conditions.

(ii) The stiffness of the linear system is obtained as the secant stiffness observed in the nonlinear motion at the instant in which the displacement attains the maximum value:

$$k_{\mathrm{s}} = \frac{f_{\mathrm{dm}}(T_{\mathrm{m}})}{u_{\mathrm{m}}(T_{\mathrm{m}})} \tag{10}$$

where u_{m} and f_{dm} are the extreme values of the displacement and the restoring force, respectively, whereas T_{m} is the period of the external force producing the maximum response in the nonlinear system. The subscript s is used to identify the linear parameters deduced from the stationary response. An equivalent linear stiffness G_{s} may be deduced for the rubber as follows:

$$G_{\mathrm{s}} = \frac{h}{A} k_{\mathrm{s}} \tag{11}$$

where h is the thickness and A the area of the rubber device.

(iii) The damping coefficient is obtained by requiring that the energies dissipated by the two systems in the two reference motions are the same:

$$W_{\mathrm{d}}^{\mathrm{L}}(T_{\mathrm{m}}^{\mathrm{L}}) = W_{\mathrm{d}}(T_{\mathrm{m}}) \tag{12}$$

where $W_{\mathrm{d}}^{\mathrm{L}}$ is the energy dissipated by the linear system and $T_{\mathrm{m}}^{\mathrm{L}}$ is the external force period producing the maximum displacement in the linear system. The damping coefficient has the expression

$$\xi_{\mathrm{s}} = \frac{W_{\mathrm{d}}(T_{\mathrm{m}})}{2\pi k_{\mathrm{s}} u_{\mathrm{s}}^2} \tag{13}$$

DOI: 10.1002/eqe

where u_s is the displacement of the linear system at the resonance condition, which is expressed by

$$u_s = \frac{f_0}{k_s} \frac{1}{2\xi_s} \tag{14}$$

By substituting Equation (14) in Equation (13), the following expression is obtained:

$$\xi_s = \frac{\pi f_0^2}{2k_s W_d(T_m)} \tag{15}$$

Four cases are considered in the analyses in order to have dynamical systems that exhibit peak displacements for external force periods of approximately 0.5 s (case *a*), 1.0 s (case *b*), 2.0 s (case *c*) and 3.0 s (case *d*). The chosen period values make it possible to study different situations equivalent to different structural systems where devices are introduced in order to reduce the seismic effects, such as frames with dissipating bracings ($T = 0.5$ s–1.0 s) or isolated structures ($T = 1.0$ s–3.0 s). In this sense it should be noted that the dynamic properties of the system do not change by modifying the area A and thickness h of the device as well as the mass m of the system while keeping the ratio A/hm constant. Here, the case *b* is obtained with a rubber layer of area $A = 78\,200\,\text{mm}^2$ and thickness $h = 10\,\text{mm}$ while considering a mass of 10^5 kg. The other cases *a* and *c* are obtained with a device area four times larger and four times smaller, respectively, whereas the area of case *d* is about eight times smaller than the area of case *b*. For each case, it was assumed that $\beta = 1$ and the maximum intensity of the external force f_0 was calibrated to provide maximum values of shear strain ($\gamma = u/h$) equal to 2.0, 1.5 and 1.0, which are the usual maximum strain values used in the design. Figure 1 describes the maximum values of displacements and restoring forces per unit mass observed in the nonlinear system (solid line) together with the results of the linear approximation (dashed line). The diagrams are in a nondimensional format obtained by dividing displacements, forces and periods by the reference values defined with the criteria indicated below. The reference displacement u_{ref} and the reference force f_{dref} are the value of the maximum displacement and the value of the maximum device force attained by the nonlinear system with the maximum external force, respectively. The reference period T_{ref} is the period at which the maximum values of the response of the nonlinear system are reached.

In general, the curves obtained for the approximated linear systems at different levels of external forces adequately describe the maximum values of displacements and forces. In the case of response curves involving small strains ($\gamma = 1.0$), the periods at which the response is extreme are almost coincident for linear and nonlinear systems. Large differences may be observed for higher strain levels ($\gamma = 1.5$ and 2.0). Away from the peak response, the dynamic response produced by the linear system is quite different from the nonlinear response: the displacements are usually overestimated and the force is underestimated. The differences are larger for case *d* (the most deformable system) and smaller for case *a* (the stiffest system). The secondary peak can obviously not be described by the linear models in all the cases studied.

The values of the equivalent stiffness and the equivalent damping ratio obtained by applying the first equivalence criterion are reported in Table I, for the four cases analysed and for different levels of strain. The numerical results show that the equivalent stiffness significantly decreases moving from case *a*, which vibrates rapidly and has a larger viscous response, to case *d*, which vibrates slowly. As a result the stiffness variation with the period of the system cannot be neglected in the seismic design, at least not from structural periods below 3 s. As expected, the equivalent

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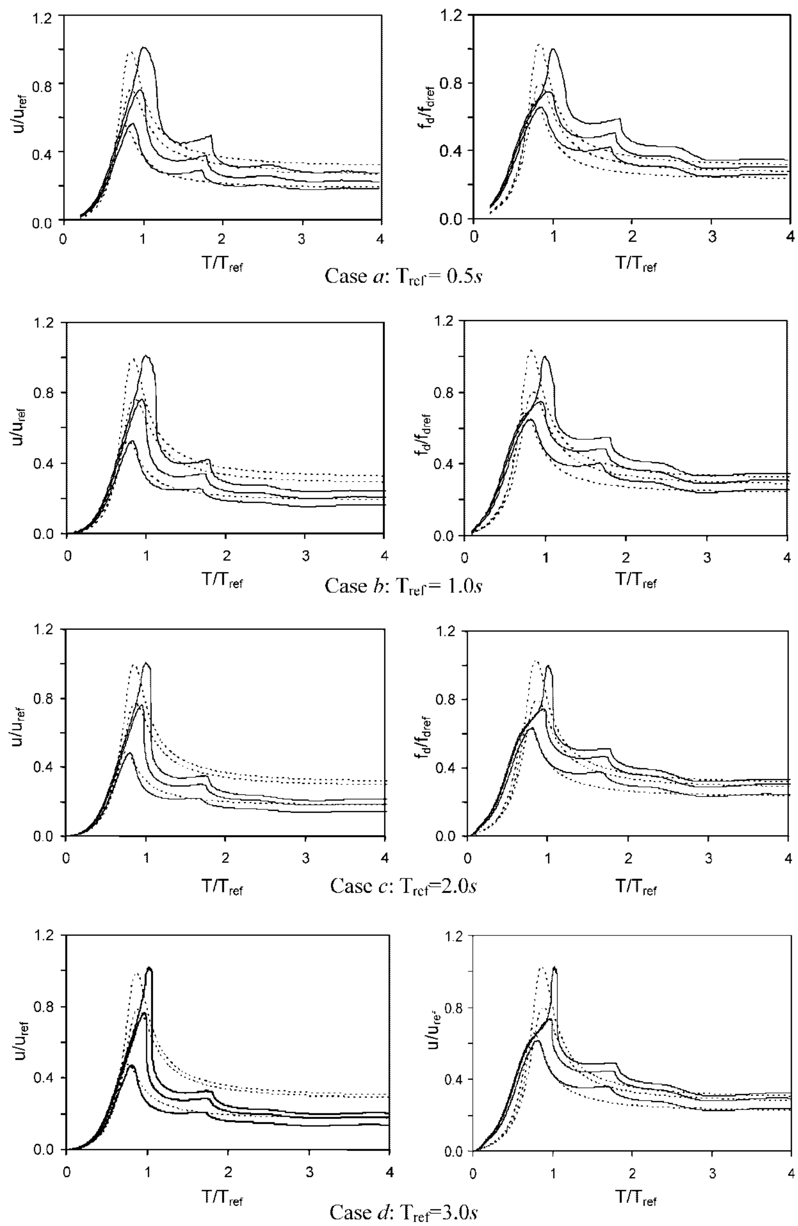

Figure 1. Maximum displacement *vs* external force period and maximum HDR force *vs* external force period diagrams obtained by harmonic analysis for three different levels of external force: nonlinear system (solid line) and linear equivalent system (dashed line).

DOI: 10.1002/eqe

Table I. Equivalent linear parameters—stationary response.

	$\gamma=2$		$\gamma=1.5$		$\gamma=1$	
	G_s (N mm^{-2})	ξ_s	G_s (N mm^{-2})	ξ_s	G_s (N mm^{-2})	ξ_s
Case *a*	0.866	0.15	0.857	0.18	1.010	0.18
Case *b*	0.753	0.16	0.746	0.19	0.939	0.18
Case *c*	0.672	0.16	0.660	0.19	0.883	0.18
Case *d*	0.654	0.15	0.634	0.18	0.859	0.18

stiffness is strongly influenced by the strain amplitude and decreases when the strain increases. Contrary to the equivalent stiffness, the equivalent damping ratio does not show a remarkable variation and the values observed are fixed between 0.15 and 0.18. This means that the dissipated energy approximately varies proportionally to the stiffness.

3.3. Transient response

The Mullins effect strongly influences the system behaviour in the initial path of the response under external force. To better analyse this case, the systems are subjected to initial conditions consisting of an initial velocity v_0, which is associated with an initial value for the kinetic energy $W_0=\frac{1}{2}mv_0^2$, while keeping the initial values of the state variables equal to zero. The equivalent linear systems are obtained by adopting rules similar to those used in the previous section. In particular:

(i) The first part of the motions of the two systems up to maximum displacement are selected as similar dynamic situations for the application of the equivalence conditions.

(ii) The stiffness of the linear system is obtained as the secant stiffness observed in the nonlinear motion at the instant in which the displacement attains the maximum value:

$$k_t=\frac{f_{\mathrm{dm}}(t_1)}{u_{\mathrm{m}}(t_1)} \tag{16}$$

where t_1 is the time at which the maximum displacement of the nonlinear system is reached. The subscript t is used to identify the linear parameters deduced from the transient response. An equivalent linear stiffness G_t is then deduced for the rubber.

(iii) The equivalent damping coefficient is obtained by requiring that the energies dissipated by the two systems in the two reference motions are the same:

$$W_{\mathrm{d}}^{\mathrm{L}}(t_1^{\mathrm{L}})=W_{\mathrm{d}}(t_1) \tag{17}$$

where t_1^{L} is the time at which the maximum displacement of the linear system is reached. The energy dissipated at the time t_1^{L} by the linear system is

$$W_{\mathrm{d}}^{\mathrm{L}}(t_1^{\mathrm{L}})=W_0-W_{\mathrm{e}}^{\mathrm{L}}(t_1^{\mathrm{L}}) \tag{18}$$

where

$$W_{\mathrm{e}}^{\mathrm{L}}(t_1^{\mathrm{L}})=\tfrac{1}{2}k_t u^2(t_1^{\mathrm{L}}) \tag{19}$$

DOI: 10.1002/eqe

Based on Equation (17) it is possible to find the coefficient ξ_t once the expression for the displacement of a linear system subjected to an initial velocity v_0 is introduced in the equation. This may be expressed in the following form:

$$u(t)=\frac{v_0}{\omega_t}\mathrm{e}^{-\xi_t\omega t}\sin\omega_t t \tag{20}$$

where $\omega_t=\sqrt{k_t(1-\xi_t^2)}$.

The four cases defined previously are considered and the initial velocity is calibrated in order to attain the same maximum displacements adopted in the previous section, corresponding to the limit shear strains $\gamma=2.0, 1.5, 1.0$. Figure 2 describes the time histories of displacement and the device force per unit mass observed in the nonlinear system (solid line) together with the results of the linear approximation (dashed line). As in the previous section, the results are presented in a nondimensional format. The same values of the reference displacement, device force and time interval are considered to simplify the comparison of the results. The curves obtained by using the linear approximated systems adequately describe maximum values of displacements and forces in the case of response curves involving small strains ($\gamma=1.0$), whereas large differences are observed for higher strain levels ($\gamma=1.5$ and 2.0). Generally, the linear systems are stiffer with respect to nonlinear systems and are not able to describe the responses after the first quarter of a cycle.

The values of the equivalent stiffness and the equivalent damping ratio are reported in Table II for the four cases analysed and for different levels of the strain.

An analysis of such parameters makes it possible to conclude that the transient behaviour of the rubber is meaningfully different from the stationary behaviour. In particular, the stiffness shows larger values that increase considerably by increasing deformation and decreasing the period of the system. This behaviour is due to the increasing importance of the Mullins effect, which depends on the strain amplitude and the strain rate. At the same time, the damping coefficients are slightly different from those obtained by the stationary behaviour and sensitively decrease by reducing deformation and the reference period.

4. SEISMIC RESPONSE

This section analyses the response of the previous four cases subjected to external inputs describing seismic events, within the range of material strains and strain rates previously considered. More specifically, seven artificial ground motions are considered. The accelerograms are generated to match the elastic response spectrum given by [16] for zone 1 and ground types B-C-E, according to the rules furnished by the same code. The code spectrum and the average spectrum of the seven accelerograms in terms of pseudo-velocity are reported in Figure 3.

In order to obtain results that are comparable with others, situations with the same average value of maximum strain equal to $\gamma_{\mathrm{d}}=1.5$ are considered. Consequently, in the four cases analysed, an area A^* and a thickness h^* are assigned to obtain the desired maximum strain $\gamma_{\mathrm{d}}=1.5$ under the input intensity considered and by maintaining their ratios constant. Table III reports the values of the area and the thickness obtained in the four cases and the values of the equivalent stiffness and the equivalent damping coefficient of the respective linear models. In particular, the parameters of the linear model related to the stationary response ($k_{\mathrm{s}}, \xi_{\mathrm{s}}$), the parameters of the linear model related to the transient response (k_t, ξ_t) and the average values of previous parameters (k_a, ξ_a) are

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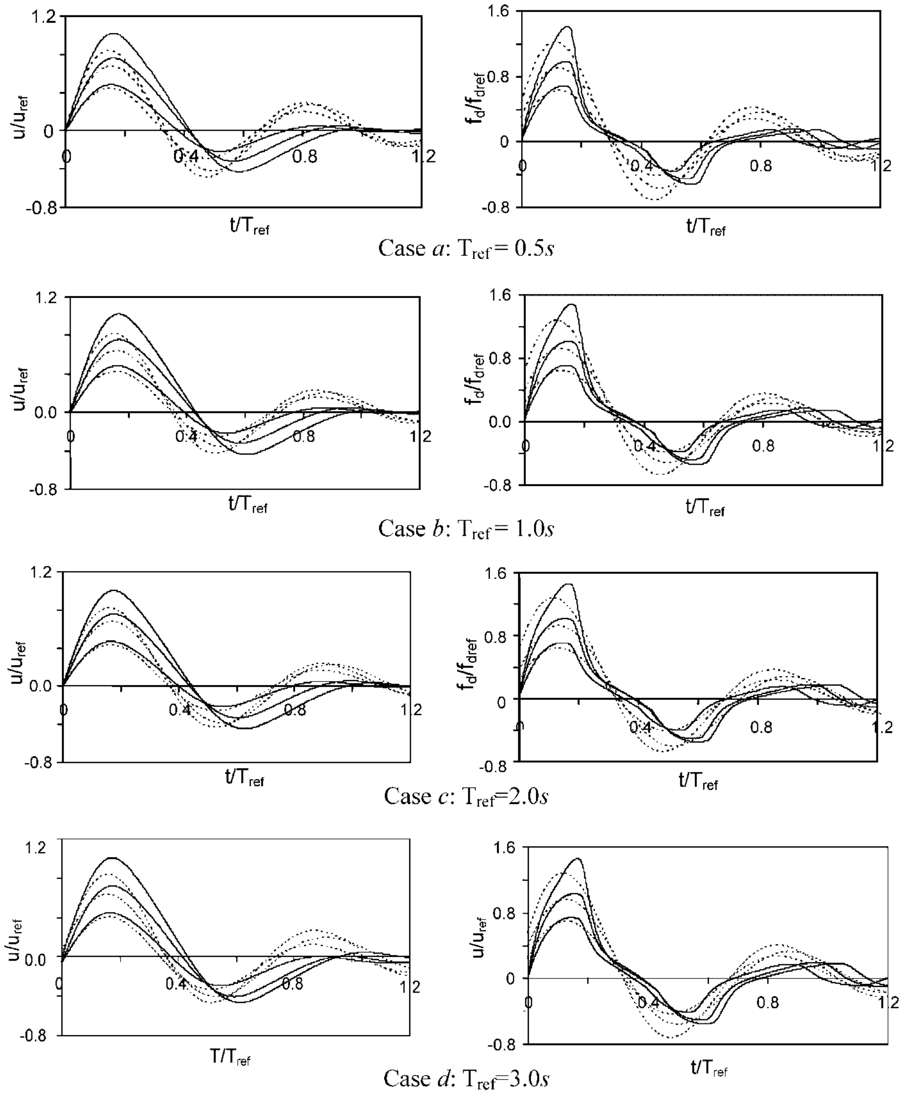

Figure 2. Displacement time histories and HDR force time histories obtained by impulsive analyses with three different levels of initial velocity: the nonlinear system (solid line) and the linear equivalent system (dashed line).

reported in the table. Stiffness values are obtained from the shear module values, as shown below:

$$k_s(T_{ref}, \gamma_d) = \frac{A^*}{h^*} G_s(T_{ref}, \gamma_d) \tag{21}$$

$$k_t(T_{ref}, \gamma_d) = \frac{A^*}{h^*} G_t(T_{ref}, \gamma_d) \tag{22}$$

Table II. Equivalent linear parameters—transient response.

	$\gamma=2$		$\gamma=1.5$		$\gamma=1$	
	G_t (N mm^{-2})	ξ_t	G_t (N mm^{-2})	ξ_t	G_t (N mm^{-2})	ξ_t
Case *a*	1.202	0.16	1.119	0.14	1.170	0.14
Case *b*	1.087	0.19	1.005	0.17	1.074	0.16
Case *c*	0.972	0.18	0.895	0.17	0.972	0.15
Case *d*	0.945	0.19	0.865	0.17	0.959	0.15

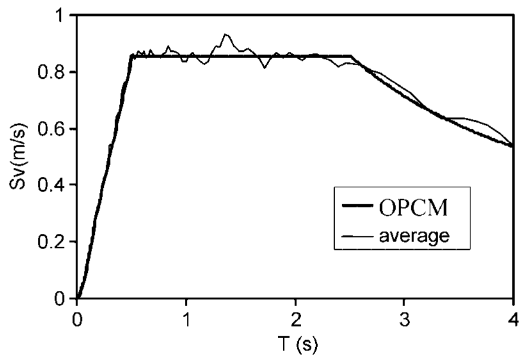

Figure 3. Pseudo-velocity spectra.

Table III. Geometric characteristics of HDR devices and equivalent linear parameters.

	HDR		Lin_s		Lin_t		Lin_a	
	h^* (mm)	A^* (mm^2)	k_s (kN mm^{-1})	ξ_s	k_t (kN mm^{-1})	ξ_t	k_m (kN mm^{-1})	ξ_m
Case *a*	14	437 920	26.80	0.18	35.00	0.14	30.90	0.16
Case *b*	40	312 800	5.83	0.19	7.86	0.17	6.85	0.18
Case *c*	85	166 180	1.29	0.19	1.75	0.17	1.52	0.18
Case *d*	125	117 300	0.6	0.18	0.81	0.17	0.7	0.175

whereas

$$k_a = \frac{k_s + k_t}{2} \tag{23a}$$

$$\xi_m = \frac{\xi_s + \xi_t}{2} \tag{23b}$$

The extreme values for the displacements and forces obtained using the nonlinear model and the three linear models are compared in Figure 4, where the maximum values for displacements and forces are reported on the abscissa axis, while the results obtained by the three linear analyses are reported on the ordinate axis. The bisecting line, shown by the dashed line, indicates when the displacements or forces obtained using the linear models coincide with those obtained from the nonlinear analysis.

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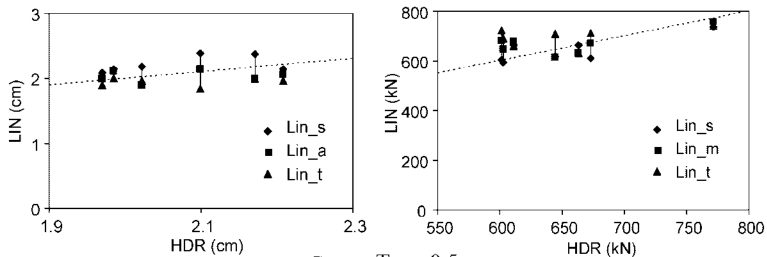
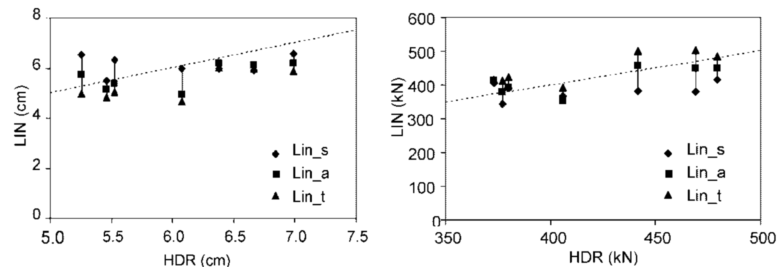
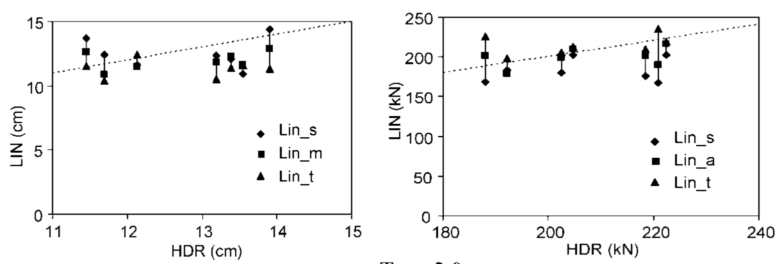


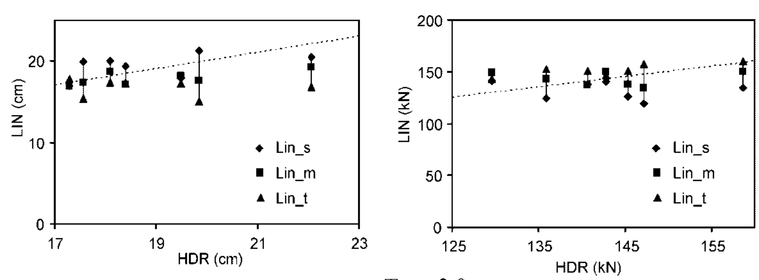
Case a: $T_{ref} = 0.5s$



Case b: $T_{ref} = 1.0s$



Case c: $T_{ref} = 2.0s$



Case c: $T_{ref} = 3.0$

Figure 4. Maximum displacements and forces for the seven ground motions considered.

It may be observed that the linear model referring to the stationary behaviour (Lin_s) shows a tendency to overestimate the displacements and underestimate the forces, whereas the linear model referring to the transient behaviour (Lin_t) has the opposite tendency. This is confirmed by the average values reported in Table IV. The results obtained show that the two equivalent models, defined according the criterion proposed, may provide the extreme values of forces and

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Table IV. Average results obtained with nonlinear and linear systems.

	Nonlinear		Lin_s		Lin_t		Lin_a	
	Displ. (mm)	Force (kN)	Displ. (mm)	Force (kN)	Displ. (mm)	Force (kN)	Displ. (mm)	Force (kN)
Case *a*	21.4	652	22.7	641	19.6	695	20.8	668
Case *b*	60.5	418	61.2	383	53.4	446	56.7	413
Case *c*	127.5	207	124.9	182	113.1	215	119.5	199
Case *d*	189.6	143	194	133	167	151	179	142

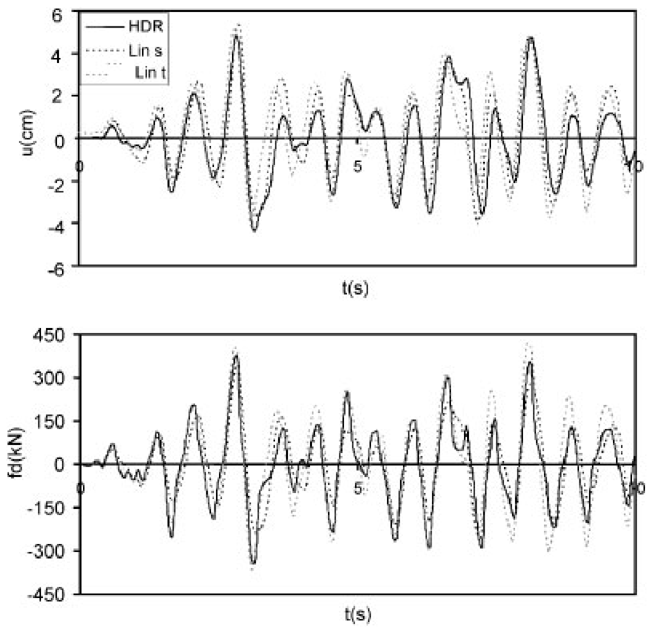

Figure 5. Displacement and force time histories related to an accelerogram applied to case *b*.

displacements and their results do not differ more than 10% with respect to those obtained with the nonlinear analysis.

It is important to observe that the linear models adopted produce good estimates of the extreme values of force and displacement but are entirely inadequate to describe the response for smaller values of deformation, different from those considered in their definition, as shown in Figure 5, for the time histories of force and displacement related to an accelerogram applied to case *b*.

5. APPROXIMATE EVALUATION OF EQUIVALENT PARAMETERS

Previous results are based on the dynamic properties of linear and nonlinear systems and showed that it is possible to obtain upper and lower bounds of the solution by means of two different linear models coherently deduced from the stationary and the transient responses. A number of problems,

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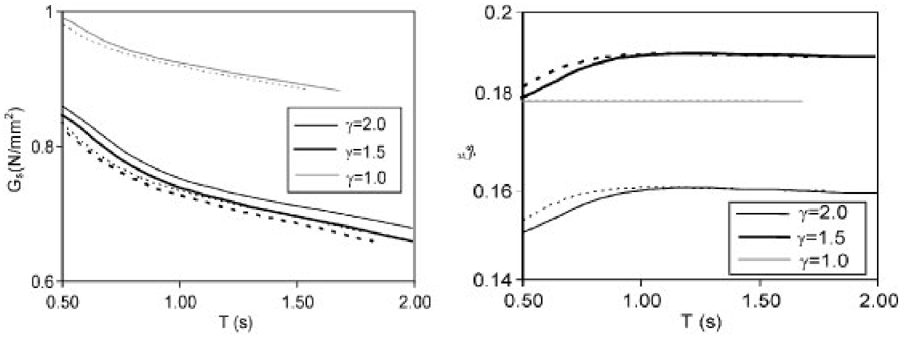

Figure 6. Diagram of G_s and ξ_s, obtained from the analysis of the dynamic systems (continuous line) and from displacement-controlled cyclic tests (dashed line) for case *b* studied.

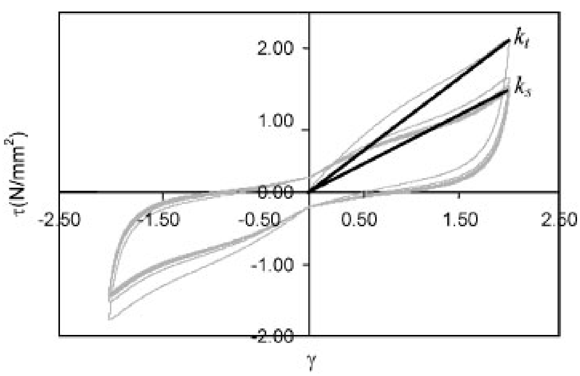

Figure 7. Evaluation of the equivalent stiffness k_s and k_t from displacement-controlled sinusoidal tests.

however, arise in practical applications. The information on the dynamical system, with respect to harmonic and impulsive responses, may be obtained from experimental tests on the system or by using appropriate numerical models to describe HDR behaviour. The former is usually quite complex and expensive, while latter is fairly cumbersome and adequate models are generally not available in practical applications.

It would be very useful to derive relationships or graphs providing equivalent parameters (G and ξ) from classical displacement-controlled tests on the devices.

With reference to the stationary behaviour, the results obtained demonstrated that at the resonance condition the differences between the stationary dynamic response and the sinusoidal displacement-controlled test at the same frequency and amplitude are very small. Thus, the method proposed by Hwang *et al*. [12], Yoshida *et al*. [13] and Hwang and Ku [14] and accepted by technical codes [16–18] provides results that are close to the method proposed by the authors for the stationary response. The only significant approximation introduced is due to the difference of periods between the nonlinear system and the equivalent linear system. An estimation of the approximation introduced is reported in Figure 6.

For the transient response, the results obtained evidenced that the differences between the first part of the transient response and the first cycle of the sinusoidal displacement-controlled test

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carried out at the same amplitude and frequency are not very large. Consequently k_t may be deduced as the ratio between the extreme force reached during the first cycle and the maximum displacement (Figure 7). On the other hand, it is very difficult to coherently deduce the parameter ξ_t from the experimental test, since the dissipated energy cannot be experimentally measured. The results given in the previous paragraphs, however, showed that the parameter ξ does not remarkably vary between the transient and stationary responses so that the estimation obtained for the stationary response may also be used for the transient response ($\xi_s \cong \xi_t = \xi$); this means that in the first part of the first cycle the energy dissipated increases proportionally to stiffness.

6. CONCLUSIONS

Current design codes suggest linear equivalent models to model HDR behaviour. These models are generally defined by considering the stationary and quasi-static behaviour of the device. They generally neglect the transient response of the material, which strongly influences the seismic behaviour of structures with HDR-based devices, since the extreme values of displacements and velocities are reached only a few times during a seismic event.

This paper proposes a method to define equivalent linear models on the basis of the behaviour of a dynamic system with restoring force furnished by HDR-based devices. In order to approximate the transient and the stationary behaviours, two linear models were introduced. A parametric analysis was carried out on S-DoF systems with different dynamic properties giving different maximum shear strains in order to evaluate the effectiveness of linear approximation in evaluating displacements, forces and natural periods.

The results obtained show that the variation of the stiffness with the system period of the system and the displacement amplitude is significant and consequently cannot be neglected. On the other hand, the equivalent damping coefficient may be considered almost a constant. Additionally, the transient behaviour of the rubber is notably different from the stationary behaviour. In particular, the stiffness values obtained are remarkably larger compared with those obtained considering a stationary behaviour in similar conditions, whereas the damping coefficients are slightly different from those obtained from stationary behaviour.

The effectiveness of the proposed models in predicting the seismic response was checked by carrying out linear and nonlinear analyses of simple one degree-of-freedom systems in the strain amplitude and strain rate fields of interest in the seismic design. The results obtained show that the two equivalent models, defined according the criterion proposed, provide the extreme force and displacement values with acceptable levels of approximation.

Finally, an approximated procedure to define linear equivalent systems from quasi-static displacement-controlled cyclic tests is proposed. Comparison with previous results, based on the dynamic response of the systems, showed that stiffness and damping may be estimated with an acceptable level of approximation.

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