Nonlinear behavior of dynamic systems with high damping rubber devices

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A B S T R A C T

The High Damping Rubber (HDR) is widely used in seismic engineering and, more generally, in the passive control of vibrations. Its constitutive behaviour is quite complex and is not simply non-linear with respect to strain but also shows a transient response during which material properties change (Mullins effect). A number of recent works were dedicated to analyzing and modelling the material behaviour. The present work intends to study the consequences of such non-linear behaviour in the dynamic response of S-DoF systems where the restoring force is provided by dissipative devices based on the HDR (structural system with dissipative bracings and isolated systems). Preliminary analyses under harmonic forces and impulsive excitations were carried out in order to separately characterize stable and transient responses. Finally, the response under seismic inputs with different intensities was studied. Results show that the Mullins effect may play an important role in the seismic response and the dynamic properties of the system change significantly for seismic events with different intensities.

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1. Introduction

In the field of seismic engineering the rubber with enhanced dissipating properties, usually known as High Damping Rubber (HDR), is extensively adopted in bearings for the seismic isolation of bridges or buildings [1], and is also used for dissipating devices in order to increase stiffness and the energy dissipation capacity of structures [2–5]. With respect to other types of damper devices, based on elasto-plastic or viscous materials, the HDR-based damper seems to be a promising energy dissipating device because no permanent strains occur after seismic events and moreover it permits dissipating energy even for small lateral displacements produced by wind or minor earthquakes.

Some difficulties in the use of this kind of dissipating device derive from its complex dynamic behaviour which, makes it difficult to evaluate the behaviour of equipped structures accurately and to give design indications. More specifically, the material behaviour is strongly non-linear and both stiffness and damping properties vary with the amplitude of strain and depend on the strain rate [6,7]. Furthermore, the presence of filler added to the natural rubber, makes the response of the HDR strain history-dependent and causes a transient behaviour in which stiffness and damping change remarkably. The phenomenon, usually known as the “Mullins effect” or “scragging”, is a consequence of the damage of the microstructure, that occurs during the process [8,9]. Recent studies [10] showed that the transient response is related to the maximum strain attained by the material and is influenced by the strain rate. The initial properties of the material may be however recovered (healing effect) and the healing times depend on the material considered and on the temperature [10]. The rubber studied in [11] showed a rapid recovery of a large part of the material stiffness even if the complete recovery may take several months. Consequently seismic analyses of structures endowed with HDR devices should be performed with virgin material properties, even if some scragging process were applied to the devices, and a model with damage should be adopted. The influence of scragging on the seismic response is also evidenced in [12] where the bidimensional response of an isolated bridge is analyzed.

The present work intends to analyze the dynamic response of single degree of freedom systems in which the restoring force is given by HDR devices, by using a unidimensional model based on virgin properties of the rubber and including the scragging phenomenon, previously developed by the authors [11]. The aim of these numerical analyses is to evidence characteristic aspects which can be of interest in the structural design under seismic actions and which cannot be described by simpler models, such as linear visco-elastic or elasto-plastic, usually used to simulate the HDR-based devices behaviour [13,14]. The analyses consider a range of the shear strain from 0.0 to 2.0, which seismic dissipating devices usually undergo. Three different ratios between mass and stiffness have been considered in order to study the rubber response in different dynamic situations, spanning from vibrations with long period, which furnish information about isolated systems, to vibrations with short period, which furnish...
information about the rubber response in dissipating braces inserted within deformable frames.

In particular, Section 3 investigates the harmonic behaviour of the dynamical systems subjected to sinusoidal forces. The results refer to the stable, post-transient, response and furnish information regarding the influence of non-linear behaviour of the HDR on the dynamic response of the system, once the Mullins effect is over.

The following section is otherwise oriented to highlight the influence of the Mullins effect that influences the initial part of the dynamic response. For this purpose, the system behaviour under an impulsive initial input is studied.

Finally, the last section is dedicated to the analysis of the system subjected to seismic inputs. The analyses show that the Mullins effect may influence and change the system response in the case of similar seismic events (similar frequency content and maximum displacement attained). Furthermore, the influence of all the non-linear phenomena on the response under seismic events of different intensities is analyzed in order to show how the dynamic properties of the system change by varying the input intensity.

2. Dynamical system

2.1. HDR model

The model used to simulate the HDR response is a rheological unidimensional model able to describe the transient response of the rubber, depending on both the strain rate and the maximum strain attained, as evidenced by experimental tests reported by Fig. 1. In the model the state of the material is furnished by the shear strain \( \gamma \), defined as the ratio between the shear displacement and the thickness of the rubber, and by a set of internal variables \( \alpha_i \) describing the inelastic response and the Mullins effect. The tangential stress may be derived from the free energy per unit volume \( \phi_d(\gamma; \alpha_i) \) by the relation

\[
\tau_d = \frac{\partial\phi_d}{\partial\gamma}
\]

whereas the dissipated power per unit volume \( w_d \) may be obtained from the derivative with respect to the internal variables (repeated indexes denote summation, superposed dot denotes time derivative)

\[
w_d = \frac{\partial\phi_d}{\partial\alpha_i} \dot{\alpha}_i.
\]

The stress deriving from a strain history may be determined once the initial state and the process \( \eta = \dot{\gamma} \) are known, on the basis of the nonlinear functions \( g_i(\gamma, \eta; \alpha_i) \), which describe the evolution of the internal variables:

\[
\begin{bmatrix}
\dot{\gamma} \\
\dot{\alpha}_i
\end{bmatrix} = \begin{bmatrix}
\eta \\
g_i(\gamma, \eta; \alpha_i)
\end{bmatrix}.
\]

(3)

The expressions of the free energy and those of the evolution functions adopted in the following analyses are reported and commented in the Appendix.

2.2. S-DoF system

The S-DoF (Single-Degree of Freedom) dynamical system considered consists of a mass \( m \) and an HDR-based dissipating device that furnishes the restoring force. It is assumed that a linear relation, defined by a constant \( c \), exists between the mass displacement \( u \) and the shear strain of the device rubber \( \gamma = cu/h \), where \( h \) is the thickness of the rubber layer. The value of \( c \) depends on the geometry of the connection between the device and mass. The restoring force per unit mass \( f_d \) can be expressed in the form

\[
f_d = \frac{CA}{m} \tau_d
\]

(4)

where \( A \) is the area of the HDR layer in the device. The state of the system is consequently described by the vector \( x = [u, v; \alpha_i] \) where \( v \) is the velocity of the mass. The evolution law has the following form:

\[
\dot{x} = \begin{bmatrix}
\dot{u} \\
\dot{v} \\
\dot{\alpha}_i
\end{bmatrix} = \begin{bmatrix}
f_d \left( c \frac{u}{h}, \frac{v}{h}, \frac{\dot{\gamma}}{h}; \alpha_i \right) + f_e \\
\frac{CA}{mh} \tau_d(\gamma, \eta; \alpha_i) + f_e \\
g_i(\gamma, \eta; \alpha_i)
\end{bmatrix} = A(x)
\]

(5)

where \( f_e \) is the external force per unit mass. It may be useful to observe that the following non-linear results may also be extended to cases different from those considered here. The equation of motion can be rewritten by remembering that \( u = \gamma h/c \) and by dividing each term by the thickness \( h \). The equation assumes the form

\[
\dot{x} = \begin{bmatrix}
\dot{\gamma} \\
\dot{\eta} \\
\dot{\alpha}_i
\end{bmatrix} = \begin{bmatrix}
\frac{CA}{mh} \tau_d(\gamma, \eta; \alpha_i) + \frac{f_e}{h} \\
g_i(\gamma, \eta; \alpha_i) + \frac{f_e}{h}
\end{bmatrix} = A(x)
\]

(6)

consequently the same strain history may be observed in all those cases where the two ratios \( CA/mh \) and \( f_e/h \) did not vary (e.g. non-linear response does not vary by doubling \( f_e \) if both \( A \) and \( h \) are also doubled).
3. Harmonic analysis

In this section, a harmonic analysis is carried out in order to investigate the nonlinear response of the system subjected to periodic external force with varying frequency and amplitude.

In the tested range of external actions the system shows a transient response and attains a stable behaviour after a certain number of cycles. The response related to an external action with period $T$ is considered to be stable at the instant $t$ when the difference between the state history observed in the last period and the state history observed in the previous period is sufficiently small. More precisely, once the two state functions $x_1(t) = x(t - T + s)$ and $x_2(t) = x(t - 2T + s)$ have been defined on the same interval $s \in [0, T]$ and a positive real constant $\epsilon$ has been chosen, the response is considered to be stable if $|x_1 - x_2| < \epsilon$. In the application it was assumed that $\|y(s)\| = \max\{y(s), s \in [0, T]\}$.

Having the aim to characterize the response under seismic events acting on the system where the state variables are null initially, the analyses were performed by assuming $x = 0$ for $t = 0$. The external force has the expression

$$f_e(t) = f_0 \sin(2\pi t/T)$$

where $f_0$ is the amplitude of the force per unit mass and $T$ is its period. In particular three cases corresponding to three values of stiffness are considered in the analyses. The intermediate case $b$ was obtained by a rubber layer with an area $A = 78200 \text{ mm}^2$ and a thickness $h = 10 \text{ mm}$; a mass of $10^5 \text{ kg}$ was considered. The other cases $a$ and $c$ were obtained with a device area four times larger and smaller, respectively. Such values were chosen in order to obtain dynamical systems that show the displacement peak for external force periods of about 0.5 s (case $a$), 1.0 s (case $b$) and 2.0 s (case $c$). The chosen stiffness values make it possible to study different situations referable to different structural systems where devices are introduced in order to reduce seismic effects, like frames with dissipating bracings ($T = 0.5$–1.0 s) or isolated structures ($T = 1.0$–2.0 s). The external force periods considered in the analyses span from 0.3 to 4.0 s, which is the range of interest in seismic response. Finally, the constant $c = 1$ was assumed in the analyses.

For each case, the maximum intensity of the external force $f_{\text{om}}$ was calibrated to provide a maximum value of shear strain ($\gamma = u/h$) equal to 2.0, which is an usual maximum strain value in the design since larger strains cause a strongly hardening behaviour of the rubber. Other results were also evaluated for smaller values of the maximum strain by considering different intensities of the external force, equal to 0.75$f_{\text{om}}$, 0.5$f_{\text{om}}$ and 0.25$f_{\text{om}}$. The constant $c$ of Eq. (4) was taken as being equal to 1.0.

Fig. 2 reports the maximum values of displacement ($u$), the restoring forces ($f_0$) the energy $W_d$ dissipated in a cycle and the extreme values $\Phi_r$ attained by the free energy in the periodic response. The diagrams are in a non-dimensional format and were obtained by dividing displacements, forces, energies and periods by reference values defined with the criteria indicated below. The reference displacement $u_{\text{ref}}$ and the reference force $f_{\text{ref}}$ are the value of the maximum displacement and the value of the maximum device force attained with the external force $f_{\text{om}}$. The reference period $T_{\text{ref}}$ is the period at which these maximum values are reached. The reference energy value is given by

$$E_{\text{ref}} = \frac{1}{2} f_{\text{ref}} u_{\text{ref}}^2.$$  

The results refer to case $b$, for which $f_{\text{om}} = 37 \text{ kN}$, $u_{\text{ref}} = 20 \text{ mm}$, $f_{\text{ref}} = 1.20 \text{ N kg}^{-1}$ and $T_{\text{ref}} = 1.0 \text{ s}$. By observing Fig. 2 it is evident that the main displacement peak occurs at a period ($T_{\text{m}}$) which decreases by decreasing the force intensity. This is a typical behaviour of softening systems even if the behaviour seems to become weakly hardening for the largest value of the input force, as is usual in the large strain range. The amplitude of the main peak increases non-linearly with the intensity of the input force, as may be seen in the same figure. This trend is mainly controlled by the dissipation properties of the system, which worsen for large strains.
The maximum displacement rapidly changes for periods which are slightly larger than $T_m$, although multiple responses were not observed in the cases analyzed. Predictably, unstable behaviour may be exhibited by the system for larger strains. It can be also observed that for every value of the input force the response curves have a secondary peak, whose period is about 1.8 times that of the main peak.

In order to analyze the transient response and the shape of the stable loop of the system subjected to harmonic loads a number of analyses in the time domain were reported. Specifically two periods were considered: $T_1 = T_m$ and $T_2 = 1.8T_m$ with the input force $f_{in}$. The analysis results, displacement vs time and force vs displacement, are reported by Fig. 3. In the case of $T_1 = T_m$ the displacements are approximately sinusoidal and are strongly amplified. In the case of $T_2 = 1.8T_m$ the displacement history becomes periodic but is no longer sinusoidal. In this case displacement contributions with higher frequencies arise. Similar trends were observed in the other cases $a$ and $c$, even if a number of changes occur. A comparison may be obtained from Fig. 4 where the maximum values of displacements and shear stresses ($\tau_d = f_d/A$) of the previous case $b$ are reported together with results of case $a$, which is stiffer and attains maximum values for about $T_m = 0.5$ s, and case $c$, which has a lower stiffness and attains maximum values for about $T_m = 2$ s. The main differences regard the secondary peak and stress intensity, which increase for stiffer system as a consequence of the increment in the strain rate.

In order to furnish synthetic information and to permit a numerical comparison between different situations, three parameters were chosen to describe the system response. The first parameter

$$G = \frac{f_{dm}}{u_{max}A}$$

(9)

is the ratio between the device force and displacement at the instant when the system attains the maximum displacement in the periodic response. It may be interpreted as an approximated estimation of the material stiffness (shear modulus) at the maximum displacement and is strongly dependent on the period $T_m$.

The other two parameters are derived from energy quantities and describe the dissipative properties of the system. The first parameter is the damping coefficient, which may be defined, in analogy of linear systems, as

$$\xi = \frac{W_{dm}}{4\pi \Phi_{em}}$$

(10)

where $W_{dm}$ is the dissipated energy and $\Phi_{em}$ is the peak of the energy stored in the system during a period. The previous parameter ($\xi$) furnishes information on the ratio between dissipated and stored energy and will be evaluated, as usual, for the period at which the response is at its maximum value, changing case by case.

Sometimes, a different parameter $\xi_e$ (equivalent damping coefficient) in which $\Phi_{em}$ is substituted by $E_{ref}$, was adopted to study rubber behaviour [13–15]. This latter was not used because it has no physical meaning, since $E_{ref}$ does not measures the internal energy of the material.

In order to describe the dissipation rate and, more generally, the global ability of the system to reduce the effects induced by an external input, another parameter

$$\lambda = \frac{2\pi \xi}{T_m}$$

(11)

is introduced. This was obtained by dividing the previous parameter by the duration of the time interval on which the dissipated energy is evaluated (average dissipated power). The constant term $2\pi$ was added in order to make it similar to the coefficient describing the exponential decay in linear systems. Table 1 reports the results evaluated at $T_m$ and for the cases and the external force levels considered previously.

Predictably, by comparing the results referring to the same case ($a$, $b$ or $c$), it may be observed that the shear modulus of the material strongly increases when the maximum strain decreases. On the contrary, the damping coefficient does not change significantly by reducing the external force (the dissipated energy increases in the same way as the free energy) whereas the damping rate coefficient augments as the strain decreases (the energy is dissipated more rapidly). On the other hand by comparing results...
Referring to similar strain amplitudes, it is possible to observe that by decreasing the period of the input force ($T_m$) the shear modulus slightly increases, whereas the damping coefficient may be considered constant. Considering all the cases analyzed, the values of the shear modulus span from 0.67 to 1.66, whereas the range variation of the damping coefficient is from 0.21 to 0.29.

### 4. Impulsive excitation

The Mullins effect mainly influences the system behaviour in the initial path of the response under external force. In order to analyze a dynamic situation strongly affected by the transient phenomena related to the Mullins effect, the system was subjected to initial conditions consisting of an initial velocity $v_0$, to which an initial value of kinetic energy ($E_0 = 0.5m v_0^2$) is associated, while the other state variables were taken equal to zero.

The three previously defined cases were considered and the initial velocity was calibrated in order to attain the same maximum displacement adopted in the previous section (harmonic analysis) and corresponding to the limit shear strain, $\gamma = 2.0$. Other results were also evaluated by reducing the external input and assuming the following values for the initial velocity: $0.75v_0$, $0.50v_0$, $0.25v_0$.

Fig. 5 describes the time histories of displacement and the device force for the intermediate case $b$. Here again, the results are presented in a non-dimensional format. The same values of the reference displacement, device force and time interval were adopted in order to simplify the comparison with the results of the harmonic analysis. Results of case $b$ were obtained with $v_0 = 200$ mm s$^{-1}$.

Comparing, the cyclic response and the free vibration motion corresponding to the same limit strain $\gamma = 2.0$, it is evident that the free system vibrates more rapidly at the initial stage and the maximum value of the device force is notably higher. In other words, the system is remarkably stiffer. A more direct understanding of the differences in the stable and transient response for similar values of maximum strain may be obtained by Fig. 6 where the device stress–strain graphs of the two analyses (harmonic and impulsive) are traced together for two different values of the external input multiplier. The differences are more remarkable for large strains, which are also related to a large strain rate. The graph seems to show that the transient response is also more dissipative. Here again, similar trends were observed in the other cases $a$ and $c$.

The initial behaviour of the system may be characterized on the basis of the motion observed in the initial time interval $[0, t^*]$, assuming $t^*$ as the instant at which displacement return to zero. The same three parameters used in the harmonic analysis were evaluated. In particular the stiffness $G$ was evaluated using Eq. (9), the damping coefficient was evaluated from the energy decay, likewise in linear systems. The expression is

$$\xi = \frac{\ln E(t^*) - \ln (E_0)}{2\pi \sqrt{1 - \xi^2}}$$

### Table 1

<table>
<thead>
<tr>
<th>Case</th>
<th>$\gamma$</th>
<th>$f_0$ (N kg$^{-1}$)</th>
<th>$T_m$ (s)</th>
<th>$G$ (N mm$^{-2}$)</th>
<th>$\xi$</th>
<th>$\lambda$ (s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>2.00</td>
<td>1.66</td>
<td>0.47</td>
<td>0.86</td>
<td>0.26</td>
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<td></td>
<td>1.12</td>
<td>1.24</td>
<td>0.40</td>
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<td></td>
<td>0.58</td>
<td>0.83</td>
<td>0.34</td>
<td>1.31</td>
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<td>3.10</td>
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<td></td>
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<td>0.30</td>
<td>1.66</td>
<td>0.22</td>
<td>3.78</td>
</tr>
<tr>
<td>b</td>
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<td>1.00</td>
<td>0.75</td>
<td>0.29</td>
<td>0.94</td>
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<td></td>
<td>1.04</td>
<td>0.27</td>
<td>0.82</td>
<td>0.93</td>
<td>0.24</td>
<td>1.36</td>
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<td>0.72</td>
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<td>0.22</td>
<td>1.51</td>
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<td>0.61</td>
<td>1.56</td>
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</tr>
<tr>
<td>c</td>
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<td>2.08</td>
<td>0.67</td>
<td>0.27</td>
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<tr>
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<td>0.88</td>
<td>0.23</td>
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<td>0.22</td>
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Fig. 6. Comparison between stable and transient response for maximum strain $\gamma = 2$ (a) and $\gamma = 1$ (b).

Table 2
Parameters describing the transient response

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$v_0$ (mm/s)</th>
<th>$T^*$ (s)</th>
<th>$G$ (N mm$^{-2}$)</th>
<th>$\xi$</th>
<th>$\lambda$ (s$^{-1}$)</th>
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<td>0.20</td>
<td>0.86</td>
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Table 3
Parameters describing the seismic response (case b)

<table>
<thead>
<tr>
<th>Accelerogram</th>
<th>$\gamma$</th>
<th>$F$ (kN)</th>
<th>$G$ (N mm$^{-2}$)</th>
<th>$E_i$ (kN mm)</th>
<th>$W_{d0} / E_i$</th>
<th>$W_{dM} / E_i$</th>
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<td>175</td>
<td>1.11</td>
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<td>1.04</td>
<td>557</td>
<td>0.83</td>
<td>0.17</td>
</tr>
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</table>

where

$E(t^*) = E_0 - W_d(t^*)$.  

(13)

The trend of the energy decay is shown in Fig. 7 for the studied case b and the initial velocity $v_0$. The third parameter, describing the exponential decay, was evaluated by the expression

$\lambda = \frac{\pi \xi}{t^*}$  

(14)

which is similar to the previous Eq. (11) while taking into account that, in this case, the system describes approximately half a sinusoid during the time $t^*$. Table 2 reports the results evaluated for $t^*$ and for the cases and the external force levels considered previously.

By comparing results referring to the same case (a, b or c) it may be observed that in all of the cases analyzed, especially for large strains, the values of the shear modulus related to similar strains, are significantly larger than those obtained in the harmonic case. On the other hand the values obtained for the equivalent damping coefficient are not very different from those obtained from the stable response. This means that stiffness and dissipative properties are proportionally enhanced.

Observing the dissipation rate coefficients, it is however possible to observe that the system dissipates more quickly in the transient response with respect to the stable response, especially for large values of strain (when the Mullins effect is more important).

5. Seismic excitation

The aim of this section is to study how and how much nonlinear phenomena previously analysed may affect the seismic response. In particular the influence of the accelerogram type and accelerogram intensity on the system response is investigated.

5.1. Influence of the accelerogram type

Seven ground motions were chosen from the earthquakes available on the European Strong-motion Database (http://www.isesd.cvic.ac.uk) related to a ground type C (soft soil). Each accelerogram has a different PGA and spectrum shape. Coherently with the approach of the previous sections, whose aim is to furnish a characterization of the non-linear system at particular values of the rubber strain, each accelerogram was scaled in order to obtain the maximum shear strain $\gamma = 2$. The accelerograms obtained have about the same spectral ordinate at the period of the system considered as reported by Fig. 8, with reference to the case b. Table 3 reports the values of the force, the previously defined parameter $G$, the input energy and the ratio between the dissipated energies (energy $W_{d0}$ dissipated by the damage induced by Mullins effect and the energy $W_{dM}$ dissipated by viscous phenomena) and the input energy ($E_i$).

Even if the accelerograms were calibrated to produce the same maximum displacement, the maximum forces observed are quite different from each other. Their values span from 175 kN, in the case of accelerogram 1, to 129 kN in the case of accelerogram 2. This is mainly due to the Mullins effect which makes the response dependent on the particular history considered. The two
The deformation histories are reported by Fig. 9a. It may be observed that the former exhibits the largest strain at the initial part of the history while the latter attains maximum values only after a number of lower cycles were traced. As a consequence, the stress of the former case are more strongly influenced by the Mullins effect and the force is higher, as may be observed in Fig. 9b where the stress–strain diagrams of the two cases are reported. The influence of the Mullins effect may also be analyzed by observing the trends of energy $W_{\text{M}}$ dissipated by the damage induced by the Mullins effect and energy $W_{\text{visc}}$ dissipated by viscous phenomena.

The progress of these energy contributions are reported by Fig. 9c together with the input energy $E_i$. It is evident that in the former case the damage dissipation due to the Mullins effect occurs mainly in the initial response, whereas the damage progress more uniformly in the latter case.

Obviously the values of the parameter $G$ are also different for each accelerogram and span from the values obtained with the stable response to those obtained by studying the transient behaviour.

The same maximum force trend may be observed in the other cases. In case a the forces span from 646 kN to 745 kN, whereas in case c these span from 30 kN to 39 kN.

The mean values of the force, the parameter $G$, the input energy and the ratio between the dissipated energies and the input energy, related to all the cases analyzed, are reported by Table 4. It should be noted that the part of the energy dissipated by the Mullins effect is significant in all the cases, especially in the case c and the mean values obtained of $G$ are closer to those obtained considering the transient response.
Table 4

<table>
<thead>
<tr>
<th>β</th>
<th>γ</th>
<th>F (kN)</th>
<th>G (N mm⁻¹)</th>
<th>E₁ (kN mm)</th>
<th>W₆₀/E₁</th>
<th>W₆₅/E₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75</td>
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<td>2692</td>
<td>1.106</td>
<td>3383</td>
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<tr>
<td>0.50</td>
<td>1</td>
<td>156</td>
<td>0.997</td>
<td>641</td>
<td>0.84</td>
<td>0.16</td>
</tr>
<tr>
<td>0.25</td>
<td>1</td>
<td>36</td>
<td>0.921</td>
<td>128</td>
<td>0.82</td>
<td>0.18</td>
</tr>
</tbody>
</table>

Table 5

Results for different levels of seismic input (case b)

<table>
<thead>
<tr>
<th>β</th>
<th>γ</th>
<th>F (kN)</th>
<th>G (N mm⁻¹)</th>
<th>E₁ (kN mm)</th>
<th>W₆₀/E₁</th>
<th>W₆₅/E₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.3</td>
<td>37</td>
<td>1.577</td>
<td>19</td>
<td>0.74</td>
<td>0.26</td>
</tr>
<tr>
<td>0.50</td>
<td>0.8</td>
<td>73</td>
<td>1.166</td>
<td>113</td>
<td>0.76</td>
<td>0.23</td>
</tr>
<tr>
<td>0.75</td>
<td>1.4</td>
<td>110</td>
<td>1.604</td>
<td>276</td>
<td>0.77</td>
<td>0.23</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>156</td>
<td>0.997</td>
<td>641</td>
<td>0.84</td>
<td>0.16</td>
</tr>
</tbody>
</table>

5.2. Influence of the accelerogram intensity

Finally, a comparison between solutions at different levels of peak ground acceleration was carried out in order to examine the response by varying seismic events. Calculations were repeated by considering the previous set of accelerograms scaled by the values 0.75, 0.50, 0.25. By reducing the input, different counteracting phenomena which make the difference with respect to previous analyses, may be foreseen. In fact, the stable cyclic response becomes stiffer when strain decreases. On the other hand small analyses, may be foreseen. In fact, the stable cyclic response by varying seismic events. Calculations were repeated by peak ground acceleration was carried out in order to examine the peak response by varying seismic events. Calculations were repeated by peak ground acceleration was carried out in order to examine the

Fig. 10. Modula of the scaled function versus frequency (case b).

Table 5

Results for different levels of seismic input (case b)

<table>
<thead>
<tr>
<th>β</th>
<th>γ</th>
<th>F (kN)</th>
<th>G (N mm⁻¹)</th>
<th>E₁ (kN mm)</th>
<th>W₆₀/E₁</th>
<th>W₆₅/E₁</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.3</td>
<td>37</td>
<td>1.577</td>
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<td>0.74</td>
<td>0.26</td>
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<td>73</td>
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<td>0.75</td>
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<td>276</td>
<td>0.77</td>
<td>0.23</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>156</td>
<td>0.997</td>
<td>641</td>
<td>0.84</td>
<td>0.16</td>
</tr>
</tbody>
</table>

6. Conclusions

The behaviour of the dynamic systems with HDR restoring force was studied in three steps: analysis of the response under harmonic excitation to characterize the stable behaviour; analysis under impulsive excitation to characterize the initial transient response and analysis under seismic excitation. The input level was assigned in order to characterize the behaviour at assigned values of the maximum strain and three ratios between mass and HDR device were considered to span cases of interest in structural design. Some synthetic measures of stiffness and dissipative properties were introduced to permit a comparison between the results.

The harmonic analysis revealed that the maximum displacement function exhibits a shape characteristic of softening systems in the neighbourhood of its maximum values and a secondary response peak arises for periods about 1.8 times the period at which the maximum value is reached. Global stiffness and dissipative properties depend on the input level.

The study of the initial response under impulsive excitation showed that the system vibrates more rapidly. The response is notably stiffer and more dissipative as a consequence of the Mullins effect. Stiffness and dissipative properties are proportionally enhanced.

The analysis under seismic input evidenced that the response is strongly different from the stable behaviour and remarkably depends on the particular strain history. Furthermore, the dynamic properties change with the input level and different sensitivity of the system with respect to input frequency content may be observed for accelerograms with different PGA.

As expected, the dynamics of HDR-based systems are quite complex in that all the phenomena characterizing the material behaviour, the transient response and the non-linear dependence on strain and strain rate, strongly influence the dynamic behavior of the systems and the more important design parameters, such as maximum displacements and forces. This aspect cannot be neglected in seismic design of dissipated or isolated structures and should be considered in elaborating simplified approaches used in analysis and design procedures.

Acknowledgement

The present work was financed by the Italian research grant ReLUIS project – D.P.C. (task 7).

Appendix

The following expression of the free energy per unit volume was adopted for the rubber:

\[
\varphi_d (\gamma, \xi) = (1 + \alpha_e) (1 - \alpha_1) \varphi_0 (\gamma) + \frac{E_{e1}}{2} (\gamma - \alpha_1)^2 + \frac{E_{e2}}{2} (\gamma - \alpha_2)^2 + (1 - \alpha_5) \frac{E_{e3}}{2} (\gamma - \alpha_3)^2
\]
and the expression of stress may be obtained by means of Eq. (1); it has the following form

$$\tau_d = f_e(\gamma) + E_{e1} [\gamma - \alpha_1] + E_{e2} [\gamma - \alpha_2] + \alpha_m (1 - \alpha_4)f_e(\gamma) + E_{e3} (1 - \alpha_5) [\gamma - \alpha_3]. \quad (A.2)$$

A rheological model consisting of the sum of two contributions was adopted. The former contribution is described by constitutive laws that do not change during strain history. The stress is provided by the sum of a nonlinear elastic contribution $f_e(\gamma) = d\phi_e(\gamma)/d\gamma$ and two nonlinear viscoelastic contribution described by the constants $E_{e1}, E_{e2}$ and two internal variables (dashpot strains) $\alpha_1, \alpha_2$, related to different relaxation times.

The latter furnishes the transient response (Mullins effect) which degenerates as the strain history progresses. The stress is provided by the sum of a contribution which is proportional to elastic stress $f_e(\gamma)$ ($\alpha_m$ is the proportional constant) and a viscoelastic contribution described by the constant $E_{e3}$ and the related internal variable $\alpha_3$. The other internal variables $\alpha_4$ and $\alpha_5$ increase during deformation and control the degrading response.

The evolution laws of internal variables have the following expressions

$$\dot{\alpha}_1 = \left[ \frac{|\dot{\gamma}|}{\eta_1(\gamma)} + v_1 \right] E_{e1} [\gamma - \alpha_1] \quad (A.3)$$

$$\dot{\alpha}_2 = \left[ \frac{H(1 - \alpha_2)}{\eta_2} \right] |\dot{\gamma}| + v_2 E_{e2} [\gamma - \alpha_2] \quad (A.4)$$

$$\dot{\alpha}_3 = \left[ \frac{H(1 - \alpha_3)}{\eta_3} \right] |\dot{\gamma}| + v_3 E_{e3} (1 - \alpha_3) [\gamma - \alpha_3] \quad (A.5)$$

$$\dot{\alpha}_4 = \xi_e |\dot{\gamma}| (0.5 |\gamma| - \alpha_4) \quad \text{if } \alpha_4 < 0.5 |\gamma| \quad (A.6)$$

$$\dot{\alpha}_4 = 0 \quad \text{if } 0.5 |\gamma| \leq \alpha_4 \leq 1 \quad (A.7)$$

$$\dot{\alpha}_5 = \xi_e |\dot{\gamma}| (1 - \alpha_5) \quad (A.8)$$

where $H$ denotes the Heaviside function, $H(1) = 1$ if $x > 0$ and $H(0) = 0$ if $x \leq 0$.

Previous constant $E_{e1}$, the function $f_e(\gamma)$, $\eta_1(\gamma)$ and parameters $v_1, \eta_2, v_2, \eta_3, v_3, \xi_e, \xi_e$ depend on the rubber compound. For the high damping rubber considered, the values assigned to the all numerical parameters and the expression of function $f_e(\gamma)$ are reported in Table A.1. These parameters were calibrated directly from the experimental tests results [11] and refer to a rubber compound with medium stiffness.

### Table A.1

<table>
<thead>
<tr>
<th>$f_e(\gamma) \text{ (N mm}^{-2}\text{)}$</th>
<th>$E_{e1} \text{ (N mm}^{-2}\text{)}$</th>
<th>$\eta_1 \text{ (N mm}^{-2}\text{)}$</th>
<th>$\alpha_m \text{ (N mm}^{-2}\text{)}$</th>
<th>$E_{e2} \text{ (N mm}^{-2}\text{)}$</th>
<th>$\eta_2 \text{ (N mm}^{-2}\text{)}$</th>
<th>$\alpha_m \text{ (N mm}^{-2}\text{)}$</th>
<th>$E_{e3} \text{ (N mm}^{-2}\text{)}$</th>
<th>$\eta_3 \text{ (N mm}^{-2}\text{)}$</th>
<th>$\alpha_m \text{ (N mm}^{-2}\text{)}$</th>
<th>$\xi_e$</th>
<th>$\xi_e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.029\gamma^3 + 0.082\gamma^3 + 0.29\gamma^3$</td>
<td>2.56</td>
<td>0.078</td>
<td>0.179 + 0.127(1 - $\alpha_4$) + 0.047$</td>
<td>\gamma$</td>
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<td></td>
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<tr>
<td>$\alpha$</td>
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<td>0.025</td>
<td>1.0</td>
<td>0.2</td>
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### References


