Convergence analysis of a parallel interfield method for heterogeneous simulations with dynamic substructuring

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SUMMARY

In order to predict the dynamic response of a complex system decomposed by computational or physical considerations, partitioned procedures of coupled dynamical systems are needed. This paper presents the convergence analysis of a novel parallel interfield procedure for time-integrating heterogeneous (numerical/physical) subsystems typical of hardware-in-the-loop and pseudo-dynamic tests. The partitioned method is an extension of the method originally proposed by Gravouil and Combescure which utilizes a domain decomposition enforcing the continuity of the velocity at interfaces. In particular, the merits of the new method which can couple arbitrary Newmark schemes with different time steps in different subdomains and advance all the substructure states simultaneously are analysed in terms of accuracy and stability. All theoretical results are derived for single- and two-degrees-of-freedom systems, as a multi-degree-of-freedom system is too difficult to analyse mathematically. However, the insight gained from the analysis of these coupled problems and the conclusions drawn are confirmed by means of the numerical simulation on a four-degrees-of-freedom system. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

As the costs and complexity of analysis, design and testing of structures, and machines and systems escalate, there is a need to analyse, develop and deploy products in the market as quickly as possible with first time quality and high profit margin, and to develop efficient approaches for the synthesis of such engineering systems. In view of engineering development of products and projects, several

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cyberinfrastructures with different characteristics have been developed recently. Distinguishing features are the uses of partitioned techniques [1, 2], modular and distributed simulations of multi-body systems [3, 4], real-time simulations [5], time-parallel frameworks for parallel simulations [6, 7], hardware-in-the-loop (HIL) and pseudo-dynamic (PsD) testing based on the principle of dynamic substructuring (DS) [8–10]. These environments integrate cutting-edge research from the fields of numerical analysis, linear/non-linear dynamics, multibody dynamics, digital control and system modelling.

Within the finite element (FE) technology, the element-based partitions that assign different sets of elements to different subdomains of the mesh and create an interface of shared nodes between subdomains are very popular [11]. In fact, these methods result in subdomains that are almost completely independent of each other, which can be solved separately and coupled together by imposing the continuity of some physical quantities at the shared nodes. It is evident that being able to deal with heterogeneous subsystems, these algorithms are amenable to HIL and PsD testing.

In the past decade, the finite element tearing and interconnecting (FETI) method emerged as one of the most powerful iterative solvers for elliptic problems and a most popular domain decomposition method for quasi-static structural mechanics problems [12]; it was later extended to dynamic problems [13]. In the FETI method, a given spatial domain is torn into nonoverlapping substructures where an incomplete solution of the primal field is evaluated using a direct solver, and intersubstructure field continuity is enforced via Lagrange multipliers applied at substructure interfaces. The latter gluing phase generates a small-size symmetric dual problem where the unknowns are the Lagrange multipliers, which is best solved with a preconditioned conjugate gradient algorithm. Gravouil and Combescure [14, 15] proved that imposing velocity continuity at the interface leads to a stable algorithm. In particular, they conceived a multi-time-step coupling method, labelled as the GC method, able to couple arbitrary Newmark schemes with different time steps in different subdomains. In this context, they proved that the GC method is unconditionally stable as long as all individual subdomains satisfy their own stability requirements. Moreover, they showed that for multi-time-step cases the GC method entails energy dissipation at the interface, whereas for the case of a single time step in all the subdomains the GC method is energy preserving. The energy dissipation and the computation of interface reactions at the smallest or fine time step in the mesh were considered as drawbacks by Prakash and Hjelmstad [16], who developed a variant of this method achieving energy preservation and elimination of interface reactions at the fine time step. Nonetheless, the computation of interface forces limited to the largest or coarse time step creates jumps in both free displacements and free velocities [16] of the fine time steps; this can be a serious drawback in HIL or PsD tests [17].

The GC method is very appealing for HIL, in particular, for the continuous PsD testing, as heterogeneous numerical and physical substructures can be solved with different implicit/explicit Newmark schemes in different subdomains according to their complexity and characteristics. The possibility of performing a large amount of small time steps on a reduced number of degrees of freedom (DoFs) at the laboratory, at about 1 kHz frequency, while computing a large time step on a large number of DoFs on a remote computer, is mandatory for the proper implementation of the continuous PsD technique with substructuring. In particular, it maintains the smoothness of the displacement trajectory without using any extrapolation/interpolation assumption, preserving the optimum signal/noise ratio of the continuous method. Unfortunately, the GC, as most of the available methods, is in essence a sequential staggered algorithm where the tasks in different subdomains are not concurrent [1]. Systematically, the process performing the fine time steps
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has to stop in order to wait for the process involving the coarse time step. In order to solve this problem, Pegon and Magonette [18, 19] developed and implemented an interfield parallel algorithm, the PM method, based on the GC method, but where the numerical subdomain (NS) and the physical subdomain (PS) states advance simultaneously and possibly continuously, as verified currently in a laboratory [20]. Although several studies point out the benefit of the parallelization of partitioned methods [1] in view of real-time and simulations of heterogeneous subsystems, there is still a paucity of publications devoted to this issue. All together, these represent the basic aspects of the temporal integration of coupled subsystems and are the issues that the paper explores further.

The remainder of this paper is organized as follows. In Section 2, we present the GC method and introduce the PM method and describe its formulation. The amplification matrix of the method derived for linear elastic problems together with the accuracy and stability analyses of the PM method using single- and two-DoFs linear model problems is presented in Section 3. Section 4 introduces additional model problems that confirm the results obtained in previous sections. Conclusions and future developments are drawn in Section 5. Finally, some developments linked to the symbolic evaluation of the amplification matrix and local truncation error are reported in Appendices A and B, respectively.

2. FORMULATION OF A PARALLEL INTERFIELD PROCEDURE

The semidiscrete second-order initial value problem with the initial conditions is analysed. In particular, the structure to be studied is divided into two subdomains, \( \mathcal{A} \) and \( \mathcal{B} \), respectively, to be integrated with time steps \( \Delta t_A \) and \( \Delta t_B \), with

\[
\Delta t_A = ss \Delta t_B \tag{1}
\]

where \( ss \) defines the number of substeps. Let \( M^A \) and \( M^B \) denote the symmetric positive-definite mass matrices of the two subdomains, \( R^A \) and \( R^B \) the internal resisting force vectors and \( F^A_{\text{ext}} \) and \( F^B_{\text{ext}} \) the vectors of applied forces, respectively. With this notation on hand, the equations of equilibrium on subdomain \( \mathcal{A} \) at time \( t_{n+1} \) and subdomain \( \mathcal{B} \) at time \( t_{n+j/ss} \), \( j = 1, \ldots, ss \), can be expressed as

\[
M^A \ddot{u}^A_{n+1} + R^A(u^A_{n+1}, \dot{u}^A_{n+1}) = F^A_{\text{ext},n+1} + L^A \dot{A}_{n+1} \tag{2}
\]

\[
M^B \ddot{u}^B_{n+j/ss} + R^B(u^B_{n+j/ss}, \dot{u}^B_{n+j/ss}) = F^B_{\text{ext},n+j/ss} + L^B \dot{A}_{n+j/ss} \tag{3}
\]

where the state variables \( u(t) \) are nodal quantities arising from a spatial discretization and their derivatives \( \dot{u}(t) \) and \( \ddot{u}(t) \) with respect to time \( t \) are indicated with superposed dots; \( L^A \) and \( L^B \) are the constraint matrices which express a linear relationship between the two connected boundaries [14]. In the case of an inelastic material, \( R \) depends also on internal variables that, in turn, incrementally depend on the current kinematical state of the numerical structure. In particular, for a linear elastic system with the classical damping \( R^A(u^A_{n+1}, \dot{u}^A_{n+1}) = K^A u^A_{n+1} + C^A \dot{u}^A_{n+1} \) and \( R^B(u^B_{n+j/ss}, \dot{u}^B_{n+j/ss}) = K^B u^B_{n+j/ss} + C^B \dot{u}^B_{n+j/ss} \), where \( K^A \) and \( K^B \) denote the stiffness matrices of the two subdomains, respectively, and \( C^A \) and \( C^B \) are the damping matrices of the two subdomains, respectively.

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The kinematic interface constraints between the subdomains can be expressed as

\[ L^A w^A_{n+j/ss} + L^B w^B_{n+j/ss} = 0 \]  

(4)

where, in general, \( w \) can be a displacement (\( u \)), a velocity (\( \dot{u} \)) or an acceleration (\( \ddot{u} \)).

2.1. A look at the GC method

In order to facilitate the understanding of the parallel interfield PM method, it is worthwhile to make some considerations on the progenitor staggered GC scheme [14]. It exploits a FETI domain decomposition as (2)–(4), which enforces (4) in terms of velocities

\[ L^A \dot{u}^A_{n+j/ss} + L^B \dot{u}^B_{n+j/ss} = 0 \]  

(5)

and for the sake of understanding it is visualized in Figure 1 in the special case of two subdomains \( A \) and \( B \). In detail, partitioned time integration schemes can be classified into two groups: the staggered procedures and the parallel procedures [1]. Referring again to Figure 1,

- the staggered procedure proceeds as follows:
  1. advance the solution in time on subdomain \( A \);
  2. transfer the information of subdomain \( A \) related to the interface to subdomain \( B \);
  3. advance the solution in time on subdomain \( B \);
  4. transfer the information of subdomain \( B \) related to the interface to subdomain \( A \).

- the parallel procedure proceeds as follows:
  1. advance the solution in time synchronously on both subdomains \( A \) and \( B \);
  2. exchange the link information at the end of each time step.

As a result, the GC method belongs to the class of interfield staggered procedures. Indeed, it presents a lot of advantages in a purely numerical environment where explicit/implicit Newmark schemes with different time steps in different subdomains can be exploited and where the parallelism is not the main issue; conversely, the parallelism is vital in real-time [9, 10] and continuous PsD [18–20] applications.

Figure 1. The GC method: basic staggered procedure.
2.2. The PM method

The novel parallel procedure proposed by Pegon and Magonette [18] is sketched in Figure 2. The time discretization on subdomains \( \mathcal{A} \) and \( \mathcal{B} \) is identical to the basic scheme of the GC method. However, a time step equal to \( 2\Delta t_A \) is exploited in subdomain \( \mathcal{A} \) in order to anticipate information on the subdomain \( \mathcal{B} \) at the beginning of a new time step. Considering arbitrary numeric schemes of the Newmark family with two characteristic parameters \( \beta_A, \gamma_A \) and \( \beta_B, \gamma_B \) for subdomains \( \mathcal{A} \) and \( \mathcal{B} \), respectively, the flow chart of the PM method [18] reads:

1. solve the free problem in subdomain \( \mathcal{A} \) using \( 2\Delta t_A \), thus advancing from \( t_n-1 \) to \( t_n+1 \)

\[
\tilde{M}^A \ddot{u}^A_{n+1\text{free}} = F^A_{\text{ext},n+1} - R^A (\widetilde{u}^A_{n-1}, \ddot{u}^A_{n-1}) \\
\dot{u}^A_{n+1\text{free}} = \ddot{u}^A_{n-1} + \beta_A (2\Delta t_A) \dddot{u}^A_{n-1} + \gamma_A (2\Delta t_A) \dot{u}^A_{n-1}
\]

with

\[
\tilde{M}^A = M^A + \alpha_1 K^A + \alpha_2 C^A \\
\ddot{u}^A_{n-1} = u^A_{n-1} + 2\Delta t_A \dddot{u}^A_{n-1} + (2\Delta t_A)^2 \left( \frac{1}{2} - \beta_A \right) \dddot{u}^A_{n-1} \\
\ddot{u}^A_{n-1} = u^A_{n-1} + 2\Delta t_A (1 - \gamma_A) \dddot{u}^A_{n-1} \\
\alpha_1 = \beta_A (2\Delta t_A)^2 \quad \text{and} \quad \alpha_2 = \gamma_A (2\Delta t_A)
\]

Here \( R^A (\ddot{u}^A_{n-1}, \dddot{u}^A_{n-1}) \) is the internal resisting force that for a linear system reads \( K^A \ddot{u}^A_{n-1} + C^A \dddot{u}^A_{n-1} \).

2. start the loop on ss substeps in subdomain \( \mathcal{B} \)

3. solve the free problem in subdomain \( \mathcal{B} \) at \( t_{n+j/\text{ss}} \) with \( j = 1, \ldots, \text{ss} \)

\[
\tilde{M}^B \dddot{u}^B_{n+j/\text{ssfree}} = F^B_{\text{ext},n+j/\text{ss}} - R^B (\widetilde{u}^B_{n+(j-1)/\text{ss}}, \dddot{u}^B_{n+(j-1)/\text{ss}}) \\
\dot{u}^B_{n+j/\text{ssfree}} = \ddot{u}^B_{n+(j-1)/\text{ss}} + \alpha_1 \dddot{u}^B_{n+j/\text{ssfree}} \\
\dddot{u}^B_{n+j/\text{ssfree}} = \dddot{u}^B_{n+(j-1)/\text{ss}} + \alpha_2 \dddot{u}^B_{n+j/\text{ssfree}}
\]

with

\[ \tilde{M}^B = M^B + z_1^B K^B + z_2^B C^B \]  

(15)

\[ \tilde{u}_{n+(j-1)/ss}^B = u_{n+(j-1)/ss}^B + \Delta t_B \tilde{u}_{n+(j-1)/ss}^B + \Delta t_B^2 \left( \frac{1}{2} - \beta_B \right) \tilde{u}_{n+(j-1)/ss}^B \]  

(16)

\[ \tilde{u}_{n+(j-1)/ss}^B = u_{n+(j-1)/ss}^B + \Delta t_B \left( 1 - \gamma_B \right) \tilde{u}_{n+(j-1)/ss}^B \]  

(17)

\[ x_1^B = \beta_B \Delta t_B^2 \quad \text{and} \quad x_2^B = \gamma_B \Delta t_B \]  

(18)

In the case of a PsD test, if no strain-rate effects are present in subdomain \( \mathcal{B} \), the internal resisting force \( R^B(\tilde{u}_{n+(j-1)/ss}^B) \) can be directly measured by imposing the displacement \( \tilde{u}_{n+(j-1)/ss}^B \) on the substructure.

4. interpolate the free velocity in subdomain \( \mathcal{A} \)

\[ \hat{u}_{n+j/\text{ssfree}}^A = \left( 1 - \frac{j}{\text{ss}} \right) \hat{u}_{n/\text{ssfree}}^A + \left( \frac{j}{\text{ss}} \right) \hat{u}_{n+1/\text{ssfree}}^A \]  

(19)

5. compute the Lagrange multipliers \( \Lambda_{n+j/\text{ss}} \) by solving the condensed interface problem

\[ H \Lambda_{n+j/\text{ss}} = - (L^A \hat{u}_{n+j/\text{ssfree}}^A + L^B \hat{u}_{n+j/\text{ssfree}}^B) \]  

(20)

where

\[ H = x_2^A L^A \tilde{M}^{A\text{ss}} \tilde{M}^{A\text{ss}}^{-1} \tilde{M}^{A\text{ss}} + x_2^B L^B \tilde{M}^{B\text{ss}} \tilde{M}^{B\text{ss}}^{-1} L^B \]  

(21)

6. solve the link problem in subdomain \( \mathcal{B} \) at \( t_{n+j/\text{ss}} \)

\[ \tilde{M}^B \tilde{u}_{n+j/\text{sslink}}^B = L^B \Lambda_{n+j/\text{ss}}^B \]  

(22)

\[ u_{n+j/\text{sslink}}^B = x_1^B \tilde{u}_{n+j/\text{sslink}}^B \quad \text{and} \quad \dot{u}_{n+j/\text{sslink}}^B = x_2^B \tilde{u}_{n+j/\text{sslink}}^B \]  

(23)

7. compute the kinematic quantities of subdomain \( \mathcal{B} \) at \( t_{n+j/\text{ss}} \), which is equal to the sum of free quantities (Point 3) and link quantities (Point 6)

\[ (\cdot) = (\cdot)_{\text{free}} + (\cdot)_{\text{link}} \]  

(24)

8. if \( j = \text{ss} \), then end the loop in subdomain \( \mathcal{B} \)

9. solve the link problem in subdomain \( \mathcal{A} \) using a time step \( 2 \Delta t_A \) from \( t_{n-1} \) to \( t_{n+1} \)

\[ \tilde{M}^A \tilde{u}_{n+1/\text{link}}^A = L^A \Lambda_{n+1}^A \]  

(25)

\[ u_{n+1/\text{link}}^A = x_1^A \tilde{u}_{n+1/\text{link}}^A \quad \text{and} \quad \dot{u}_{n+1/\text{link}}^A = x_2^A \tilde{u}_{n+1/\text{link}}^A \]  

(26)

10. compute the kinematic quantities of subdomain \( \mathcal{A} \) at \( t_{n+1} \) by summing the free problem (Point 1) and the link problem (Point 9)

\[ (\cdot) = (\cdot)_{\text{free}} + (\cdot)_{\text{link}} \]  

(27)

The strategy described above drives the algorithms illustrated in Figure 2. The dashed line describes the ongoing parallel process in the two subdomains. Note that the process in subdomain
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$\mathcal{A}$ is split into two independent parts, linked through the subdomain $\mathcal{B}$, which enables the parallel computation and synchronized exchange of information. Similar to the GC method [15], the PM method can be generalized to multiple subdomains implicit or explicit with two time scales, which was verified by the distributed testing of a bridge [20]. It is evident that the proposed method nicely suites the requirements of HIL and PsD testing of heterogeneous systems [9, 10, 18–20].

3. CONVERGENCE ANALYSIS OF THE PM METHOD

The PM method similar to the GC method uses a mesh partition with two different schemes on two or more subdomains. For this reason, it does not fall within the linear-multistep-method ($LMS$) framework, and the modal analysis approach to stability is inapplicable [21]. As a result, the GC method as other methods of this class was analysed through the energy approach [14, 21]. Although the PM method is spectrally stable as it will be shown in Section 3.3, the well-known energy approach [21, p. 564] is not easily extendible to this algorithm. The energy norm defined for the Newmark scheme, which is also applicable to the GC method [14], is however not monotonically decreasing when applied to the PM method. In this respect, see Figure 3 where the forward difference in the energy norm $[E_n] = E_n - E_{n-1}$ is given in relative terms with respect to $E_0$ for an SDoF system, where $E_n = (T^A(\dot{U}_n^A) + U^A(\ddot{U}_n^A)) + (T^B(\dot{U}_n^B) + U^B(\ddot{U}_n^B))$. In detail, $T^j(x) = \frac{1}{2}x^T \mathbf{A}^j x$, $U^j(x) = \frac{1}{2}x^T \mathbf{K}^j x$ and $A^j = M^j + \Delta T_j^2 (\beta_j - \gamma_j j/2) \mathbf{K}^j$ with $\Delta T_A = 2\Delta t_A$, $\Delta T_B = \Delta t_B$ and $\beta_j$ for the subdomain $\mathcal{A}$ or $\mathcal{B}$. The definition of an appropriate norm turns out to be a difficult task owing to the large number of system variables of the method. For this reason, consistency and stability properties of the PM method are examined exploiting a standard approach for linear model problems.

In PsD applications, it is typically preferred to use an implicit scheme in subdomain $\mathcal{A}$ for the NS and an explicit scheme in subdomain $\mathcal{B}$ for the PS. Thus, it is avoided to have Newton

![Figure 3. The forward difference in the energy norm on an SDoF system integrated by means of the PM method.](image-url)
iterations or correction formulations in the procedure devoted to the subdomain $B$, which require an estimate of the stiffness matrix of the PS. For the following analysis, we exploit the trapezoidal rule, i.e. $\beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$ in subdomain $A$, and the central difference method, i.e. $\beta = 0$, $\gamma = \frac{1}{2}$ in subdomain $B$. Nonetheless, as the original GC method, the PM scheme employed in this section is convergent with any Newmark scheme with $\gamma \neq \frac{1}{2}$.

3.1. Amplification matrix

Time stepping schemes applied to linear problems can normally be recast in a recursive form as

$$X_{n+1} = AX_n + L_n$$  \hspace{1cm} (28)

where $X$ is an appropriate state vector depending on the formulation of the scheme, $A$ is the amplification matrix and $L$ is the load vector that depends on external forces. The main difficulty in the characterization of the PM method was the choice of sufficient state variables. In order to evaluate the state of the PM method at time $t_{n+1}$, not only the variables at time $t_n$ but also some variables at $t_{n-1}$ are needed. This feature let the scheme to be a second-degree method, which can be analysed as proposed in Young [22, p. 486] in order to define the amplification matrix. To proceed to $t_{n+1}$, the knowledge of variables on subdomain $B$ at time $t_n$ is sufficient, whereas subdomain $A$ needs also the knowledge of variables at time $t_{n-1}$. Hence, the following state vector is considered:

$$X_n = (X_{n-1}^A \ X_n^A \ X_{n+1}^B)^T$$  \hspace{1cm} (29)

where $X_n^A$ collects the kinematic quantities of subdomain $A$

$$X_n^A = (u_n^A \ \dot{u}_n^A \ \overset{\text{free}}{\ddot{u}}_n^A)^T$$  \hspace{1cm} (30)

and $X_n^B$ collects those of subdomain $B$

$$X_n^B = (u_n^B \ \dot{u}_n^B \ \ddot{u}_n^B)^T$$  \hspace{1cm} (31)

As a result, $X_n$ has the dimension $8n^A + 3n^B$ with $n^A$ and $n^B$ the DoFs in the two subdomains. For homogeneous linear model problems, considered without damping for the sake of simplicity, (28) turns out to be

$$\begin{pmatrix} X_n^A \\ X_{n+1}^A \\ X_n^B \\ X_{n+1}^B \end{pmatrix} = A \begin{pmatrix} X_{n-1}^A \\ X_n^A \\ X_n^B \\ X_{n+1}^B \end{pmatrix} = \begin{bmatrix} 0 & I & 0 \\ \bar{A}^{AA} & A^{AA} & A^{AB} \\ \bar{A}^{BA} & A^{BA} & A^{BB} \end{bmatrix} \begin{pmatrix} X_{n-1}^A \\ X_n^A \\ X_n^B \end{pmatrix}$$  \hspace{1cm} (32)

where $I$ is the identity matrix, and $A$ is expressed using the block matrix form determined through the symbolic analysis provided in Appendix A.

For a general multiple DoF system, the state vectors (29)–(31) can be reduced by removing $\overset{\text{free}}{\ddot{u}}_n^A$. Indeed (9) and (10) show that $\overset{\text{free}}{\ddot{u}}_n^A$ has no contribution from the scheme to pass from $t_{n-1}$ to $t_n$. It also conforms to the fact that the corresponding row and column vectors are zero in $A$ except the submatrix part in $I$, which entails $\overset{\text{free}}{\ddot{u}}_n^A = \overset{\text{free}}{\ddot{u}}_n^A$. Summing up, the state vector $X_n$ reads

$$X_n = (u_{n-1}^A \ \overset{\text{free}}{u}_{n-1}^A \overset{\text{free}}{u}_n^A \ \overset{\text{free}}{u}_{n+1}^A \ \overset{\text{free}}{u}_n^A \ \overset{\text{free}}{u}_n^A \ \overset{\text{free}}{u}_n^A \ \overset{\text{free}}{u}_n^A)^T$$  \hspace{1cm} (33)

with a dimension of $7n^A + 3n^B$. 

3.2. Accuracy

To begin the time integration process, the PM algorithm needs an estimate of free velocities and accelerations at an initial time as well as quantities’ values of the previous or first step. As a result, an accuracy analysis needs to be addressed both for the initialization process and for the local truncation error as suggested in [23]. In order to obtain more seemingly convenient results along the lines of [23] and the considerations at the end of Section 3.5, we introduce an alternative form of (28)

$$\bar{X}_{n+1} = \bar{\Lambda}X_n + \bar{L}_n$$

where

$$\bar{X}_n = (u_{n-1}^A \dot{u}_{n-1}^A \ddot{u}_{n-1}^A \Delta t_A \ u_n^A \dot{u}_n^A \ddot{u}_n^A \ u_n^B \dot{u}_n^B \ddot{u}_n^B \Delta t_A)^T$$

$$\bar{\Lambda} = \bar{M}\bar{M}^{-1}, \quad \bar{L}_n = \bar{M}L_n$$

$$\bar{M} = \text{block diagonal } [I \ I \ \Delta t_A I \ I \ I \ \Delta t_A I \ I \ I \ \Delta t_A I]$$

with $I$ being the identity matrix. In particular, $\Delta t_A$ has been introduced in the acceleration terms of (35). The load vector $\bar{L}_n$ in (34) is not included in the following analysis, assuming that the power of the loading error term of its approximation is greater than the order of accuracy of the method.

3.2.1. Local truncation error. The local truncation error $\bar{\tau}_n$ is defined as

$$\bar{\tau}_n = \bar{\tau}(t_n) = \bar{\Lambda}X(t_n) - \bar{X}(t_{n+1})$$

where $\bar{X}(t_n)$ is the corresponding exact solution of the state vector $X_n$ at $t_n$. The exact solution of the free velocity, $\bar{u}_n^A(t_n)$ can be derived as follows. From (27),

$$\dot{u}_n^A(t_n) = \dot{u}_n^A(t_n) - \dot{u}_n^A(t_n)$$

where $\dot{u}_n^A(t_n)$ is the exact solution of the link velocity. Rewriting (25) at $t_n$ and considering the exact solution of $L^A T A_n$, one obtains

$$\tilde{M}^A \ddot{u}_n^A(t_n) = F_n^A$$

where $F_n^A(t_n)$ is the exact interface force. From the local equilibrium of subdomain $\mathcal{A}$, $F_n^A(t_n) = M^A \ddot{u}_n^A(t_n) + K^A u_n^A(t_n)$ and (26) entails

$$\dot{u}_n^A(t_n) = \frac{1}{2} \ddot{u}_n^A(t_n)$$

Substituting (41) into (39), one obtains

$$\dot{u}_n^A(t_n) = \frac{1}{2} \ddot{u}_n^A(t_n)$$

where an attentive reader can note that the exact solution of the free velocity $\dot{u}_n^A(t_n)$ depends on both the integration parameters and the time step.
With regard to the SDoF model problem described at length in Section 4.1, from symbolic calculations, one obtains

\[
\bar{\tau}_n = O(\Delta t_A^3) \quad \text{and} \quad \bar{\tau}_n = O(\Delta t_A^2) \quad (43)
\]

for \( ss = 1 \) and 2, respectively. Moreover, for the Two-DoF system described in Section 4.2, one obtains

\[
\bar{\tau}_n = O(\Delta t_A^3) \quad \text{and} \quad \bar{\tau}_n = O(\Delta t_A^2) \quad (44)
\]

for \( ss = 1 \) and 2, respectively. The results of (43) and (44) are expressed at length in Appendix B. Moreover, we discovered that the order of accuracy of \( \bar{\tau}_n \) for \( ss > 2 \) is identical to the one for \( ss = 2 \). The global error in Section 3.5 also confirms this conclusion.

### 3.2.2. Initialization error

From (29), one observes that two starting points on subdomain \( \mathcal{A} \) are necessary to initialize the PM method. To obtain the additional point on subdomain \( \mathcal{A} \), one may use the GC method for the first step followed by the PM method in the remaining steps, i.e. [18]

\[
X_0^A, X_0^B \rightarrow X_1^A, X_1^B \quad \text{through the GC method} \quad (45)
\]

The starting condition (45), denoted by \( S_1 \) in the sequel, entails the following initialization error:

\[
\bar{e}_2 = \bar{X}_2 - \bar{X}(2\Delta t_A) = \bar{A}X_1 - \bar{X}(2\Delta t_A) \quad (46)
\]

For the SDoF model problem, the initial condition of the exact solution entails \( u(0) = d_0, \dot{u}(0) = v_0 \). Taking

\[
X_0^A = \{d_0, v_0, v_{f.0}, a_0\}^T, \quad X_0^B = \{d_0, v_0, a_0\}^T \quad (47)
\]

with \( v_{f.0} = v_0 \) and \( a_0 \) determined by the initial equilibrium condition, the time integration advances for one step by the GC method. Then, we evaluate the errors at the end of the second coarse time step according to (46). When \( ss = 1 \) and 2, one obtains, respectively

\[
\bar{e}_2 = O(\Delta t_A^3) \quad \text{and} \quad \bar{e}_2 = O(\Delta t_A^2) \quad (48)
\]

For the two-DoF model problem, the initial condition of the exact solution reads \( u_1(0) = d_{1.0}, u_2(0) = d_{2.0}, \dot{u}_1(0) = v_{1.0} \) and \( \dot{u}_2(0) = v_{2.0} \). Then

\[
X_0^A = \{d_{1.0}, d_{2.0}, v_{1.0}, v_{2.0}, v_{f,1.0}, v_{f,2.0}, a_{1.0}, a_{2.0}\}^T, \quad X_0^B = \{d_{2.0}, v_{2.0}, a_{2.0}\}^T \quad (49)
\]

with \( v_{f,1.0} = v_{1.0}, v_{f,2.0} = v_{2.0} \) and \( a_{1.0}, a_{2.0} \) determined by the initial equilibrium condition. When \( ss = 1 \) and 2, one obtains, respectively

\[
\bar{e}_2 = O(\Delta t_A^3) \quad \text{and} \quad \bar{e}_2 = O(\Delta t_A^2) \quad (50)
\]

The results of \( \bar{e}_2 \) for \( ss > 2 \) is similar to the one for \( ss = 2 \). As a result, when \( ss = 1 \), \( \bar{e}_2 \) is at least third-order accurate; conversely, \( \bar{e}_2 \) is at least first-order accurate when \( ss > 1 \). The detailed results of \( \bar{e}_2 \) can be found in [24].

Another possibility for the initialization procedure is to compute the ‘−1’ point on subdomain \( \mathcal{A} \) backward using the Taylor series expansion of the exact solution, e.g.

\[
X_{-1}^A = X_0^A - \Delta t_A \ddot{X}_0^A + \frac{1}{2} \Delta t_A^2 \dddot{X}_0^A \quad (51)
\]

The initialization error based on the starting condition (51), denoted by \( S_2 \), is similar to that based on \( S_1 \), of which the analysis can also be found in [24].
3.3. Stability

The stability of the PM method is investigated hereafter through the spectral analysis technique, which assumes $\mathbf{F}_{\text{ext}}(t) = 0$ [25].

With regard to an SDoF model problem, the absolute values of the eigenvalues of the amplification matrices are plotted in Figure 4 with respect to $\Omega_B = \omega_B \times \Delta t_B$ employing various values of $s s$. The number of nonzero eigenvalues is found to be 6. Among them, only one pair of the complex conjugate eigenvalues are the principal eigenvalues [25, p. 100], whereas the remaining ones are spurious. The spurious ones are the so-called physically spurious eigenvalues [26, p. 139], which result from the misapplication of the initial values to the ordinary differential equation being solved as the initial value problem. For the PM scheme, we impose six initial values on the problem during the initialization, namely four in the subdomain $\mathcal{A}$ and two in the subdomain $\mathcal{B}$: that is, $\mathbf{u}^A_1, \dot{\mathbf{u}}^A_1, \mathbf{u}^A_0, \dot{\mathbf{u}}^A_0, \mathbf{u}^B_0$ and $\dot{\mathbf{u}}^B_0$ when the starting condition $S_2$ is adopted. Nonetheless, the differential equation being solved is only of second order, giving rise to four spurious eigenvalues. In detail, the eigenvalues $\lambda_i$ ($i = 1, \ldots, 10$) of the amplification $\bar{\mathbf{A}}$ for the SDoF model problem in the limit $\Delta t_A = 0$ read

$$
\begin{bmatrix}
1 & 1 & 1 & -\frac{b_1 - 2\sqrt{1 - b_1}}{2 + b_1} & -\frac{b_1 + 2\sqrt{1 - b_1}}{2 + b_1} & 0 & 0 & 0 & 0
\end{bmatrix}
$$

(52)

Figure 4. $|\lambda_i|$ of the SDoF problem for the PM method when $b_1 = 10$: (a) $s s = 1$; (b) $s s = 2$; (c) $s s = 20$; and (d) $s s = 1000$. 

if \( s = 1 \) and
\[
\begin{bmatrix}
1 & 1 & 1 & -1 & -8b_1 - \sqrt{-256 - 256b_1 + b_1^4} & -8b_1 + \sqrt{-256 - 256b_1 + b_1^4} \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
(53)

if \( s = 2 \). It can be shown that \( |\lambda_5| \) and \( |\lambda_6| \) are equal to unit in (52) and are strictly less than 1 unless \( b_1 = m_A/m_B = K_B/K_A \), as defined in Section 4.1, is either 0 or tends to \( +\infty \) in (53). Linearly independent eigenvectors exist for the given nonzero eigenvalues in (52) and (53). In general, we verified that \( \bar{A} \) has linearly independent eigenvectors for all the nonzero eigenvalues. As a result, the spectral radius of \( \bar{A} \), \( \max_{i} |\lambda_i(\bar{A})| \leq 1 \), suffices the stability. The stability limit is \( \Omega_B = 2 \) when \( ss > 1 \). When \( ss = 1 \), the stability limit is higher, which is found to be related to the system parameters \( b_1 \). Nevertheless, the scheme is always stable when \( \Omega_B < 2 \). The same behaviour has been observed for the GC method.

For the two-DoF model problem, the number of initial values provided by the integrator is \( 4n_A + 2n_B = 10 \), whereas the differential equations have only a \( 2 \times 2 = 4 \) dimension. Thus, 6 of the 10 nonzero eigenvalues are spurious. Again the stability condition is \( \Omega_B = \omega_B \times \Delta t_B < 2 \) in Figure 5, where \( r = m_A/m_B \).

![Figure 5](image-url)

Figure 5. \( |\lambda_i| \) of the two-DoF model problem for the PM method with \( r = 1 \):
(a) \( s = 1 \); (b) \( s = 2 \); (c) \( s = 20 \); and (d) \( s = 50 \).
The global algorithm has the same stability limit as the explicit subdomain where the central difference method is adopted. Hence, we conclude that the PM method preserves the same property as the GC method, i.e. it is stable if the stability requirement in each subdomain is satisfied.

3.4. Numerical damping ratio and relative period error

The numerical damping ratio induced by the PM method on the SDoF model problem is depicted in Figure 6, where \( \Omega_1 = 2\sqrt{(k/m)\Delta t_A} \) defines the nondimensional frequency of the whole system, \( 2\Delta t_A \) being the times the angular frequency of the emulated system. In detail, for the physically undamped system, the principal complex conjugate eigenvalues of the amplification matrix can be expressed in the following form [25]:

\[
\lambda_{1,2} = B \pm iC = e^{-\xi\tilde{\omega}\Delta t_A \pm i\tilde{\omega}\Delta t_A}
\]

where \( \xi \) is the numerical damping ratio and \( \tilde{\omega} \) is known as the numerical angular frequency. One can observe from Figure 6(a) that the equivalent damping generated at the shared node is modest even for larger number ss of substeps. In addition, the variation in mass and stiffness through \( b_1 \) slightly modifies the interface damping as depicted in Figure 6(b).

Figure 7 shows the numerical damping ratio on the two-DoF model problem and a comparison with the SDoF model problem. No significant variation in damping is obtained examining the first or the second mode, whereas the damping results are lower with respect to the one of the SDoF system: this trend is confirmed also for other model problems commented later. Several spurious eigenvalues being present, \( \xi \) was also evaluated through the results of simulations. In particular, we used the expression

\[
e^{-\xi\tilde{\omega}t_n} = \sqrt{\tilde{E}_n/\tilde{E}_0}
\]

where the right-hand side is known as the amplitude decay at time \( t_n \), which is equal to the square root of the ratio between the energies \( \tilde{E} \) at \( t_n \) and \( t_0 \), respectively. In detail, \( \tilde{E} \) is defined as the sum of the potential and kinematic energies in both subdomains. The \( \xi \) values estimated using (54) and (55) are denoted by eigenvalues and simulation, respectively, in Figure 7(b). It is found that the evaluation with both methods closely agrees.

---

Figure 6. Numerical damping ratio of the PM method for the SDoF model problem: (a) \( b_1 = 1 \) and (b) ss = 50.
Figure 7. Numerical damping ratio of the PM method for the two-DoF model problem: \( r = 1, ss = 50 \).

Figure 8. Relative period error of the PM method for the SDoF model problem with \( ss = 50 \).

The relative period error \((\bar{T} - T)/T = \sqrt{(k/m)/\bar{\omega}} - 1\), which defines another dispersion measure of the PM method, is plotted versus \( \Omega_1 \) in Figure 8. Its tendency is typical of implicit methods and the relative period error is greater than the one exhibited by the trapezoidal rule.

3.5. Convergence

With regard to the convergence of the PM method, the global error \( \bar{e}_n \) at \( t_n \) is affected by both the local truncation error and the initialization error,

\[
\bar{e}_n = \bar{\Lambda}^{n-2} \bar{e}_2 + \sum_{i=2}^{n-1} \bar{\Lambda}^{n-(i+1)} \bar{\tau}_i \tag{56}
\]

with

\[
\bar{e}_n = \bar{\bar{e}}(t_n) = \bar{X}_n - \bar{\bar{X}}(t_n) \tag{57}
\]

According to the Lax equivalence theorem [27, p. 40], a method will be convergent of the order \( k \), if it is stable and consistent of the order \( k \), i.e. \( \bar{\tau}_i = O(\Delta t^{k+1}_\Lambda) \) and \( \bar{e}_2 = O(\Delta t^{k}_\Lambda) \). In detail, for the PM method, \( k = 2 \) when \( ss = 1 \) and \( k = 1 \) when \( ss > 1 \).
Both Figures 9 and 10 show the global error $|e_n| = |X_n - X(t_n)|$ versus $\Delta t_A$ in a logarithmic scale for the SDoF model problem, $X(t_n)$ being the corresponding exact solution of the state vector $X_n$ at $t_n$. The initial condition is chosen to be $d_0 = 1$ and $v_0 = 0$. The simulations confirm the analytical results: the PM scheme is second-order accurate for $ss = 1$ and first-order accurate for $ss > 1$. The numerical evaluation of the global error $|e_n|$ versus $\Delta t_A$ was also performed for the two-DoF system, for which the initial values were $d_{1,0} = 0, d_{2,0} = 1$ and $v_{1,0} = v_{2,0} = 0$.

Figure 9. Global error of the SDoF model problem for the PM method with $ss = 1$: (a) $b_1 = 0.1$ and (b) $b_1 = 10$.

Figure 10. Global error of the SDoF model problem for the PM method with $ss \geq 2$: (a) $b_1 = 0.1$ with $ss = 2$; (b) $b_1 = 10$ with $ss = 2$; (c) $b_1 = 0.1$ with $ss = 20$; and (d) $b_1 = 10$ with $ss = 20$.
Figure 11. Global error of the two-DoF model problem \((r = 1)\) for the PM method (a) with \(ss = 1\); (b) with \(ss = 2\); (c) with \(ss = 20\); and (d) with \(ss = 50\).

As illustrated in Figure 11, the PM scheme is found to be second-order convergent when \(ss = 1\) and first-order convergent when \(ss > 1\).

The starting condition \(S_1\) is adopted in the simulation above. We verified that adopting the starting condition \(S_2\) the scheme provided similar results [24].

To feature more the spectral properties of the PM method, we express the discrete solution as [28]

\[
\begin{align*}
    u_n^j &= \sum_{i=1}^{m} c_i \lambda_i^n, \\
    \dot{u}_n^j &= \sum_{i=1}^{m} \bar{c}_i \lambda_i^n, \\
    \ddot{u}_n^j &= \sum_{i=1}^{m} \tilde{c}_i \lambda_i^n
\end{align*}
\]

(58)

where \(\lambda_i\)'s are the nonzero eigenvalues of the amplification matrix, \(c_i, \bar{c}_i\) and \(\tilde{c}_i\) serve as generic constants and \(j\) represents the subdomain \(A\) or \(B\). Figure 12 shows the \(|\lambda_i|\)'s and \(|c_i|\)'s of \(\dot{u}_n^A\), \(\ddot{u}_n^A\) on the SDoF model problem with \(b_1 = 10\), \(ss = 50\). The number of nonzero eigenvalues plotted in the figure is 6; among them, \(\lambda_2\) and \(\lambda_3\) are the principal complex conjugate eigenvalues with the relevant constants being different from zero at \(\Omega_1 \to 0\), whereas the others are spurious ones with the relevant constants being zero at \(\Omega_1 \to 0\). In detail, a spurious real eigenvalue \(|\lambda_1| = 1\) always exists. Figure 12 shows how \(c_1 \neq 0\) and \(c_1 = \bar{c}_1 = 0\) for \(\Omega_1 \neq 0\). This spectral behaviour is responsible for a noncontinuity of the subdomain displacements at the interface for large time steps. Conversely, the remaining spurious eigenvalues do not entail this behaviour being less
than 1. Moreover, for \( \ddot{u}_A \) the constants relevant to \( \lambda_5 \) and \( \lambda_6 \) do not vanish for \( \Omega_1 \to 0 \), and the same phenomenon is observed for \( \ddot{u}_B \). As a result, if we analyse the local truncation error 

\[ \tau_n = A\bar{X}(t_n) - X(t_{n+1}) \] 

first-order lower errors are found in the terms related to the acceleration compared with the ones related to displacement and velocity. We verified numerically that those are the effects of \( \lambda_5 \) and \( \lambda_6 \). As the moduli of \( \lambda_5 \) and \( \lambda_6 \) are less than the principal ones, these contributions will damp out and eventually will not affect the global convergence; on this basis, we introduced the alternative state vector \( \bar{X}_n \) and the amplification matrix \( \bar{A} \) in (34).

4. REPRESENTATIVE NUMERICAL SIMULATIONS

Owing to the complexity of the PM method, the convergence analysis presented in Section 3 was applied to single- and two-DoF model problems. The insight gained from these analyses and the conclusions drawn from the relevant numerical investigations are confirmed hereafter with the results provided by three- and four-DoF systems as well. Clearly, the characteristics of all model problems examined are provided in accordance.

4.1. Partitioned single-DoF system

The single-DoF mass–spring emulated system considered here is split into two parts as illustrated in Figure 13. The total mass \( m \) and the total stiffness \( k \) are the sum of the mass and the stiffness
of each part, i.e.

\[ m = m_A + m_B, \quad k = k_A + k_B \]  

(59)

The idea is to compare various cases keeping the natural frequency \( f \) of the emulated system unchanged. As a result, for the numerical analysis carried out in Sections 3.3–3.5 we assume

\[ f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = 1 \text{Hz}, \quad \frac{m_A}{m_B} = b_1 \quad \text{and} \quad \frac{k_A}{k_B} = \frac{1}{b_1} \]  

(60)

4.2. Partitioned two-DoF system

A two-DoF mass–spring emulated system can be split into two parts as depicted in Figure 14. The natural frequencies \( f_1 \) and \( f_2 \) of the emulated system are left unchanged. As a result, for the different cases examined we assume

\[ f_1 = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = 1 \text{Hz}, \quad f_2 = \sqrt{3}f_1 = \sqrt{3} \text{Hz} \quad \text{and} \quad \frac{m_A}{m_B} = r \quad \text{with} \quad m_A + m_B = m \]  

(61)

4.3. Partitioned four-DoF system

The four-DoF emulated system considered can be split into two parts as shown in Figure 15. The displacement vector of subdomain \( \mathcal{A} \) is chosen to be \( \mathbf{U}_A = \{u_2, u_3, u_4\}^T \), whereas the displacement

![Figure 13. Partitioned single-DoF system.](image1)

![Figure 14. Partitioned two-DoF system.](image2)
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Figure 15. Partitioned four-DoF system.

The vector of subdomain $\mathcal{B}$ is chosen to be $U_B = \{u_1, u_2\}^T$. The mass matrices of each subdomain are

\[
M_A = m \begin{bmatrix}
r & 0 & 0 \\
1+r & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\quad \text{and} \quad
M_B = m \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 1+r \\
\end{bmatrix}
\]  \hspace{1cm} (62)

whereas the stiffness matrices of each subdomain are

\[
K_A = 10^6 \begin{bmatrix}
35.468 & -32.555 & 9.692 \\
-32.555 & 32.262 & -5.578 \\
9.692 & -5.578 & 7.414 \\
\end{bmatrix}
\quad \text{and} \quad
K_B = 10^6 \begin{bmatrix}
12.400 & -6.600 \\
-6.600 & 3.900 \\
\end{bmatrix}
\]  \hspace{1cm} (63)

where the condensed mass $m = 20000$ kg for each DoF and $r = m_A/m_B$ with $m = m_A + m_B$. As a result, the natural frequencies for the emulated system read

\[
f_1 = 0.47\text{Hz}, \quad f_2 = 2.90\text{Hz}, \quad f_3 = 4.18\text{Hz} \quad \text{and} \quad f_4 = 9.47\text{Hz}
\]  \hspace{1cm} (64)

Figure 16 shows the results of the PM method for the case $r = 1$; the time step is chosen such that $2\pi f_1 \Delta t_A = 0.025$ and the substep number $ss = 50$. The results provided by the GC method are also provided using the $2\pi f_1 \Delta t_A = 0.05$ and the same substep number. Again, the results provided by the GC method and the PM method are very close to each other. In addition, the numerical damping developed at the interface of all model problems considered is depicted in Figure 17: one can observe that the damping is very small in all cases.

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Figure 16. Four-DoF system: (a) displacement response and (b) velocity response.

Figure 17. Numerical damping ratio of the four-DoF model problem of the PM method with $r = 1$ and $ss = 50$. 

5. CONCLUSIONS

The convergence of an interfield parallel procedure for time-integrating heterogeneous (numerical/physical) subsystems typical of HIL and PsD tests was analysed in this paper. Using this partitioned method, one can divide the whole structure into a subdomain where an implicit Newmark method can be used and a subdomain where an explicit Newmark method can be adopted. Moreover, the time step in one subdomain can be ss times that of the other one. This provides the possibility of synchronizing the computations in the two subdomains according to numerical or physical requirements. As a result, this method can be implemented not only for parallel simulations of numerical systems but also for HIL and continuous PsD testing.

The new coupling method was shown to be conditionally stable, as the stability of the explicit subdomain determines the stability of the emulated problem. As soon as $\Delta t_B$ satisfies the stability condition, a rising of ss does not have any impact on the stability. Regarding the accuracy, the scheme is still second-order accurate when ss is equal to 1, but it is first-order accurate when ss is greater than 1, typical of partitioned schemes. The numerical damping ratio that is determined by the energy dissipated at the interface is rather limited and similar when the number of substeps is different except when it is unity, which corresponds to a nondissipative case. Compared with the GC method, the PM method exhibits an accuracy which is related to $2\Delta t_A$ instead of $\Delta t_A$, and the numerical analysis shows that it results to be less dissipative than its progenitor GC method.

Based on the results of this study, it is highly likely that any dissipative time integrators employed in the proposed interfield method will be successful and a first attempt is ongoing. Finally, the extension of the partitioned method to non-linear problems requires further study.

APPENDIX A: SYMBOLIC EVALUATION OF THE AMPLIFICATION MATRIX $A$

The procedure hereafter describes the symbolic evaluation of the amplification matrix $A$ defined by (32). For subdomain $A$,

$$X_{n+1}^A = X_{n+1}^A + X_{n+1}^A$$  \hspace{1cm} (A1)

and according to (32)

$$X_{n+1}^A = \bar{A}_{\text{free}}^{AA}X_{n+1}^A + A_{\text{free}}^{AA}X^A_n + A_{\text{free}}^{AB}X^B_n$$  \hspace{1cm} (A2)

$$X_{n+1}^A = \bar{A}_{\text{link}}^{AA}X_{n+1}^A + A_{\text{link}}^{AA}X^A_n + A_{\text{link}}^{AB}X^B_n$$  \hspace{1cm} (A3)

Regarding the free problem in subdomain $A$, (6) and (7) entail

$$\bar{A}_{\text{free}}^{AA} = \begin{bmatrix} I + x_1^A Z_1 & 2\Delta t_A (I + x_1^A Z_1) & 0 & (1 - \frac{1}{2} - \beta_A) (2\Delta t_A)^2 (I + x_1^A Z_1) \\ x_2^A Z_1 & I + x_2^A 2\Delta t_A Z_1 & 0 & (1 - \gamma_A) 2\Delta t_A I + x_2^A (1 - \frac{1}{2} - \beta_A) (2\Delta t_A)^2 Z_1 \\ x_2^A Z_1 & I + x_2^A 2\Delta t_A Z_1 & 0 & (1 - \gamma_A) 2\Delta t_A I + x_2^A (1 - \frac{1}{2} - \beta_A) (2\Delta t_A)^2 Z_1 \\ Z_1 & 2\Delta t_A Z_1 & 0 & (1 - \beta_A) (2\Delta t_A)^2 Z_1 \end{bmatrix}$$  \hspace{1cm} (A4)

with $Z_1 = -\tilde{M}^{-1} K^A$,

$$A_{\text{free}}^{AA} = 0 \quad \text{and} \quad A_{\text{free}}^{AB} = 0$$  \hspace{1cm} (A5)
Equation (A5) derives from the fact that the predictors of free problem are based only on \( X_{n-1}^A \) as shown in (9), (10) and have no relationship with the quantities in subdomain \( B \): this is the very reason why the parallel scheme is not staggered. For a substep of subdomain \( B \), it holds that

\[
X_{n+j/ss_{\text{free}}}^B = \tilde{A}_{\text{free}}^{BA} X_{n-1}^A + A_{\text{free}}^{BA} X_n^A + A_{\text{free}}^{BB} X_{n+(j-1)/ss}^B \tag{A6}
\]

\[
X_{n+j/ss_{\text{link}}}^B = \tilde{A}_{\text{link}}^{BA} X_{n-1}^A + A_{\text{link}}^{BA} X_n^A + A_{\text{link}}^{BB} X_{n+(j-1)/ss}^B \tag{A7}
\]

Considering the free problem in subdomain \( B \), (12)–(14) imply

\[
A_{\text{free}}^{BB} = \begin{bmatrix}
I + \alpha_1 B Z_2 & \Delta t_B (I + \alpha_1 B Z_2) & (\frac{1}{2} - \beta_B) \Delta t_B^2 (I + \alpha_1 B Z_2) \\
\alpha_2 B Z_2 & I + \alpha_2 B \Delta t_B Z_2 & (1 - \gamma_B) \Delta t_B I + \alpha_2 B (\frac{1}{2} - \beta_B) \Delta t_B^2 Z_2 \\
Z_2 & \Delta t_B Z_2 & (\frac{1}{2} - \beta_B) \Delta t_B^2 Z_2
\end{bmatrix} \tag{A8}
\]

with \( Z_2 = -\tilde{M}_{ss}^{-1} K_{\text{free}} B \).

\[
\tilde{A}_{\text{free}}^{BA} = 0 \quad \text{and} \quad A_{\text{free}}^{BA} = 0 \tag{A9}
\]

Equation (A9) share a similar reasoning of (A5). From the solution of the Lagrange multipliers \( A_{n+j/ss} \) (20) via (19), it follows that

\[
A_{n+j/ss} = -\tilde{H}^{-1} \left( \left( 1 - \frac{j}{ss} \right) L^A \dot{u}_{\text{free}}^A + \left( \frac{j}{ss} \right) L^A \ddot{u}_{n+1/\text{free}}^A + L^B \ddot{u}_{n+j/ss_{\text{free}}}^B \right) \tag{A10}
\]

where

\[
\dot{u}_{n+1/\text{free}}^A = [A_{\text{free}}^{AA}]_{3,:} X_{n-1}^A \quad \text{and} \quad \ddot{u}_{n+j/ss_{\text{free}}}^B = [A_{\text{free}}^{BB}]_{2,:) X_{n+(j-1)/ss}^B \tag{A11}
\]

Here \([A]_{ij}\) indicates the \( i \) row of the matrix \( A \). \( \tilde{A}_{\text{free}}^{AA} \) and \( A_{\text{free}}^{BB} \) are given by (A4) and (A8), respectively. Solving the link problem in subdomain \( B \) by (22), one obtains

\[
\ddot{u}_{n+j/ss_{\text{link}}}^B = \left( 1 - \frac{j}{ss} \right) Z_3 \ddot{u}_{\text{free}}^A + \left( \frac{j}{ss} \right) Z_3 [A_{\text{free}}^{AA}]_{3,:} X_{n-1}^A + Z_4 [A_{\text{free}}^{BB}]_{2,:) X_{n+(j-1)/ss}^B \tag{A12}
\]

where \( Z_3 = -\tilde{M}_{ss}^{-1} L^{B^T} \tilde{H}^{-1} L^A \) and \( Z_4 = -\tilde{M}_{ss}^{-1} L^{B^T} \tilde{H}^{-1} L^B \). Equation (A12) can be formulated into

\[
\ddot{u}_{n+j/ss_{\text{link}}}^B = \left( 1 - \frac{j}{ss} \right) [A_{\text{link}}^{AA}]_{3,:} X_{n}^A + \left( \frac{j}{ss} \right) [A_{\text{link}}^{AA}]_{3,:} X_{n-1}^A + [A_{\text{link}}^{BB}]_{3,:) X_{n+(j-1)/ss}^B \tag{A13}
\]

where

\[
[A_{\text{link}}^{AA}]_{3,:) = [0 \ 0 \ Z_3 \ 0] \tag{A14}
\]

\[
[A_{\text{link}}^{AA}]_{3,:) = \begin{bmatrix}
\alpha_2 A Z_3 Z_1 \\
\alpha_2 A Z_3 + 2 \alpha_2 A \Delta t A Z_3 Z_1 \\
2 (1 - \gamma_A) \Delta t A Z_3 + 4 \alpha_2 A (\frac{1}{2} - \beta_A) \Delta t_B^2 Z_3 Z_1
\end{bmatrix} \tag{A15}
\]

\[
[A_{\text{link}}^{BB}]_{3,:) = \begin{bmatrix}
\alpha_2 B Z_4 Z_2 \\
\alpha_2 B \Delta t_B Z_4 Z_2 \\
(1 - \gamma_B) \Delta t_B Z_4 + \alpha_2 B (\frac{1}{2} - \beta_B) \Delta t_B^2 Z_4 Z_2
\end{bmatrix} \tag{A16}
\]
From (23), the link problem in subdomain $\mathcal{B}$ is completely characterized by

$$X^B_{n+j/\text{ss}} = \left(1 - \frac{j}{\text{ss}}\right)A^{BA}_{0\text{link}}X^A_n + \left(\frac{j}{\text{ss}}\right)A^{BA}_{\text{sslink}}X^A_{n-1} + A^{BB}_{j\text{link}}X^B_{n+(j-1)/\text{ss}}$$  \hspace{1cm} (A17)

where

$$A^{BA}_{0\text{link}} = \begin{bmatrix} \alpha^B_1[A^{BA}_{0\text{link}}]_{3,:} \\ \alpha^B_2[A^{BA}_{0\text{link}}]_{3,:} \\ [A^{BA}_{0\text{link}}]_{3,:} \end{bmatrix}, \quad A^{BA}_{\text{sslink}} = \begin{bmatrix} \alpha^B_1[A^{BA}_{\text{sslink}}]_{3,:} \\ \alpha^B_2[A^{BA}_{\text{sslink}}]_{3,:} \\ [A^{BA}_{\text{sslink}}]_{3,:} \end{bmatrix}, \quad \text{and} \quad A^{BB}_{j\text{link}} = \begin{bmatrix} \alpha^B_1[A^{BB}_{j\text{link}}]_{3,:} \\ \alpha^B_2[A^{BB}_{j\text{link}}]_{3,:} \\ [A^{BB}_{j\text{link}}]_{3,:} \end{bmatrix}$$  \hspace{1cm} (A18)

Adopting (A7), we obtain

$$\tilde{A}^{BA}_{j\text{link}} = \left(\frac{j}{\text{ss}}\right)A^{BA}_{\text{sslink}} \quad \text{and} \quad A^{BA}_{j\text{link}} = \left(1 - \frac{j}{\text{ss}}\right)A^{BA}_{0\text{link}}$$  \hspace{1cm} (A19)

From (24), the kinematic quantities of subdomain $\mathcal{B}$ at $t_{n+j/\text{ss}}$ read

$$X^B_{n+j/\text{ss}} = \tilde{A}^{BA}_jX^A_{n-1} + A^{BA}_jX^A_n + A^{BB}_jX^B_{n+(j-1)/\text{ss}}$$  \hspace{1cm} (A20)

where

$$\tilde{A}^A_j = A^{BA}_{j\text{link}}, \quad A^A_j = \tilde{A}^{BA}_{j\text{link}} \quad \text{and} \quad A^B_j = \tilde{A}^{BB}_{j\text{link}} + A^{BB}_{j\text{link}}$$  \hspace{1cm} (A21)

From the recursion of (A20) with $j = \{1, \ldots, \text{ss}\}$, we obtain the kinematic quantities of subdomain $\mathcal{B}$ at $t_{n+1}$:

$$X^B_{n+1} = \tilde{A}^{BA}_nX^A_{n-1} + A^{BA}_nX^A_n + A^{BB}_nX^B_n$$  \hspace{1cm} (A22)

where

$$\tilde{A}^{BA} = \sum_{j=1}^{\text{ss}} (A^A_j)^{\text{ss}-j}\tilde{A}^A_j, \quad A^A = \sum_{j=1}^{\text{ss}} (A^A_j)^{\text{ss}-j}A^A_j \quad \text{and} \quad A^B = (A^B_j)^{\text{ss}}$$  \hspace{1cm} (A23)

Equation (A23) provides the solution of the block matrices in $A$ defined by (32). Solving the link problem of subdomain $\mathcal{A}$ using (25) and the Lagrange multipliers $A_{n+1}$ using (20) for $j = \text{ss}$, one obtains

$$\ddot{u}^A_{n+1\text{link}} = Z^A_5\dot{u}^A_{n+1\text{free}} + Z^A_6\dot{u}^B_{n+1\text{free}}$$  \hspace{1cm} (A24)

where $Z^A_5 = -\tilde{M}^{A^{-1}}L^AT\tilde{\nabla}^{-1}L^A$ and $Z^A_6 = -\tilde{M}^{A^{-1}}L^AT\tilde{\nabla}^{-1}L^B$. Substituting (A11) for $j = \text{ss}$, one obtains

$$\ddot{u}^A_{n+1\text{link}} = Z^A_5[\tilde{A}^{AA}_3]_{3,:}X^A_{n-1} + Z^A_6[A^A_{j\text{free}}]_{2,:}X^B_{n+(\text{ss}-1)/\text{ss}}$$  \hspace{1cm} (A25)

or

$$\ddot{u}^A_{n+1\text{link}} = [\tilde{A}^{AA*}_3]_{4,:}X^A_{n-1} + [A^{AB*}_4]_{4,:}X^B_{n+(\text{ss}-1)/\text{ss}}$$  \hspace{1cm} (A26)
where

\[
[A_{\text{link}}^{AA^*}]_{4,:} = \begin{bmatrix}
2A_2^4 Z_2 Z_1 \
2A_2^4 \Delta t_A Z_2 Z_1 \
2A_2^4 (1/2) A_2^T Z_2 Z_1 \
0
\end{bmatrix}
\]

\[
[A_{\text{link}}^{AB^*}]_{4,:} = \begin{bmatrix}
2B_2^4 Z_0 Z_2 \
2B_2^4 \Delta t_B Z_0 Z_2 \
2B_2^4 (1/2) B_2^T Z_0 Z_2 \
0
\end{bmatrix}
\]

Then from (26), the *link problem* in subdomain \( \mathcal{A} \) reads

\[
X_n^{A} = \overline{A}_{\text{link}}^{AA^*} X_{n+1,\text{link}}^{A} + \overline{A}_{\text{link}}^{AB^*} X_{n+(ss-1)/ss}^{B}
\]

where

\[
\overline{A}_{\text{link}}^{AA^*} = \begin{bmatrix}
2A_2^4 & 2A_2^4 & 0 \\
2A_2^4 & 0 & 2A_2^4 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\overline{A}_{\text{link}}^{AB^*} = \begin{bmatrix}
0 & 2B_2^4 & 0 \\
0 & 0 & 2B_2^4 \\
2B_2^4 & 0 & 0 \\
\end{bmatrix}
\]

In order to obtain \( X_{n+(ss-1)/ss}^{B} \), we perform the recursion of (A20) with \( j = 1, \ldots, ss-1 \). Then, one obtains in (A3)

\[
\overline{A}_{\text{link}}^{AA} = \overline{A}_{\text{link}}^{AA^*} + \sum_{j=1}^{ss-1} \overline{A}_{\text{link}}^{AB^*} B^B_j \overline{A}_{\text{link}}^{AB^*} A_j^{BA}
\]

\[
\overline{A}_{\text{link}}^{AA} = \overline{A}_{\text{link}}^{AB^*} \sum_{j=1}^{ss-1} \overline{A}_{\text{link}}^{AB^*} A_j^{BA}
\]

From (27), the kinematic quantities of subdomain \( \mathcal{A} \) at \( t_{n+1} \) are solved. For (32), it holds that

\[
\overline{A}_{\text{link}}^{AA} = \overline{A}_{\text{link}}^{AA^*} + \overline{A}_{\text{link}}^{AA}, \quad \overline{A}_{\text{link}}^{AB} = \overline{A}_{\text{link}}^{AB^*} + \overline{A}_{\text{link}}^{AB}
\]

As a result, we obtain the complete symbolic expression of the block matrices of \( A \) defined by (32).

**APPENDIX B: RESULTS RELEVANT TO THE LOCAL TRUNCATION ERROR**

The results of the local truncation error accuracy analysis described in Section 3.2.1 are presented in detail here. In order to infer the correct exponent of \( \Delta t_A \) in the entries of \( \tilde{\tau}(t_n) \) in (38), we expand \( \tilde{\tau}(t_n) \) through the Taylor series

\[
\tilde{\tau}_n = \tilde{\tau}(t_n) = \tilde{\tau}_{0,n} + \tilde{\tau}_{1,n} \Delta t_A + \tilde{\tau}_{2,n} \Delta t_A^2 + O(\Delta t_A^3)
\]

From the identity matrix in \( A \), see (32), it follows that the first \( 3 \times n_A \) components in \( \tilde{\tau}_n \) are zero; for simplicity, these zeros are omitted in the expression of \( \tilde{\tau}_n \).

Considering the SDoF system described in Section 4.1, assume the general starting condition \( u(0) = d_0, \dot{u}(0) = v_0 \) and denote \( \omega = \sqrt{k/m} \). When \( ss = 1 \), the following terms are obtained:

\[
\tilde{\tau}_{0,0} = \tilde{\tau}_{1,1} = \tilde{\tau}_{2,2} = [0, 0, 0, 0, 0, 0, 0, 0]^T
\]
whereas if $ss = 2$

$$\bar{\tau}_{n, 0} = \bar{\tau}_{n, 1} = [0, 0, 0, 0, 0, 0, 0]^T$$

$$\bar{\tau}_{n, 2} = \begin{cases} 0, & \frac{km_A m_B (d_0 \omega \sin[\omega t_n] - v_0 \cos[\omega t_n])}{m (m_A + 4m_B)^2}, \\ 0, & \frac{km_A m_B (d_0 \omega \sin[\omega t_n] - v_0 \cos[\omega t_n])}{m (m_A + 4m_B)^2}, \\ 0, & \frac{km_A m_B (d_0 \omega \sin[\omega t_n] - v_0 \cos[\omega t_n])}{m (m_A + 4m_B)^2}, \\ \vdots & \vdots \end{cases}^{T} \quad \text{(B3)}$$

Considering the two-DoF system described in Section 4.2 and the starting condition $u_1(0) = d_{1,0}$, $u_2(0) = d_{2,0}$. $\dot{u}_1(0) = v_{1,0}$, $\dot{u}_2(0) = v_{2,0}$, when $ss = 1$ the terms of the Taylor series of the local truncation error (B1) read

$$\bar{\tau}_{n, 0} = \bar{\tau}_{n, 1} = \bar{\tau}_{n, 2} = [0, 0, 0, 0, 0, 0, 0, 0, 0]^{T} \quad \text{(B4)}$$

Conversely, when $ss = 2$ the resulting truncation error reads

$$\tau_{n, 0} = \tau_{n, 1} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^{T}$$

$$\begin{align*}
\tau_{n, 2} & = \begin{cases} 0, 0, & \frac{kr(\omega_1 ((d_{1,0} + d_{2,0}) \sin[\omega_1 t_n] + 3\sqrt{3}(d_{1,0} - d_{2,0}) \sin[\omega_2 t_n]) - (v_{1,0} + v_{2,0}) \cos[\omega_1 t_n] + 3(v_{1,0} - v_{2,0}) \cos[\omega_2 t_n])}{2m (4+r)^2}, \\ 0, 0, & \frac{kr(\omega_1 ((d_{1,0} + d_{2,0}) \sin[\omega_1 t_n] + 3\sqrt{3}(d_{1,0} - d_{2,0}) \sin[\omega_2 t_n]) - (v_{1,0} + v_{2,0}) \cos[\omega_1 t_n] + 3(v_{1,0} - v_{2,0}) \cos[\omega_2 t_n])}{2m (4+r)^2}, \\ 0, & \frac{kr(\omega_1 ((d_{1,0} + d_{2,0}) \sin[\omega_1 t_n] + 3\sqrt{3}(d_{1,0} - d_{2,0}) \sin[\omega_2 t_n]) - (v_{1,0} + v_{2,0}) \cos[\omega_1 t_n] + 3(v_{1,0} - v_{2,0}) \cos[\omega_2 t_n])}{2m (4+r)^2}, \\ \vdots & \vdots \end{cases}^{T} \quad \text{(B5)}
\end{align*}$$

where $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$.

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